

Reconnection of Superfluid Vortex Bundles

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Using the vortex filament model and the Gross-Pitaevskii nonlinear Schroedinger equation, we show that bundles of quantized vortex lines in He II are structurally robust and can reconnect with each other maintaining their identity. We discuss vortex stretching in superfluid turbulence and show that, during the bundle reconnection process, kelvin waves of large amplitude are generated, in agreement with the finding that helicity is produced by nearly singular vortex interactions in classical Euler flows.

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Concentrated, tubular vortical regions are commonly observed in turbulent flows. These structures are important in the dynamics; for example, they enhance mixing and diffusion. The role which they play in determining turbulence statistics (spectra, intermittency) is the subject of much study [1]. If the axes of tubular vortex structures are interpreted as the skeleton of turbulence, then the knottedness of the axes characterizes the topology, and vortex reconnections [2] are the critical events which change this topology. This idealized picture becomes reality if one moves from ordinary fluids to superfluids (liquid ^4He and ^3He , atomic Bose-Einstein condensates, and neutron stars). The reason is that in superfluids quantum mechanics constrains any rotational motion to ultrathin vortex filaments whose core radius and circulation are fixed, and turbulence takes the form of a tangle of such discrete filaments [3]. Recent experimental, theoretical, and numerical studies have revealed similarities between ordinary turbulence and superfluid turbulence [4], such as the same Kolmogorov $-5/3$ energy spectrum [5] and the same temporal decay law of the vorticity [6].

This Letter is concerned with vortex reconnections and vortex stretching in superfluid turbulence. The importance in He II of vortex reconnections and its scaling laws [7] was first appreciated by Schwarz [8], who developed the reconnecting vortex filament model; later, the existence of reconnections was confirmed by Koplik and Levine [9] using the nonlinear Schroedinger equation (NLSE) model. Vortex reconnections are associated with the dissipation of superfluid kinetic energy in the limit of absolute zero [10], either directly [11] or via a kelvin wave cascade [12]. Recently, individual vortex reconnections have been detected in experiments [13]. As recognized by Procaccia and Sreenivasan [14], the occurrence of the $-5/3$ law in superfluid turbulence is surprising if one notices that vortex stretching, usually recognized as an important mechanism to transfer energy across length scales, is absent because the radius of the superfluid vortex core is fixed. A possible solution of the puzzle is that some filaments are organized in vortex bundles, which have indeed been noticed in the most recent numerical simulations of superfluid turbulence

[15]. Clearly, stretching occurs if the relative position of vortex strands within a bundle changes during the evolution.

In a first set of numerical experiments we use the model of Schwarz [8]. Let the space curve $\mathbf{s}(\xi, t)$ represent a vortex filament where ξ is arclength and t is time. The self-induced velocity \mathbf{v}_{si} of the vortex at \mathbf{s} is

$$\mathbf{v}_{\text{si}} = \frac{d\mathbf{s}}{dt} = -\frac{\kappa}{4\pi} \oint \frac{(\mathbf{x} - \mathbf{z})}{|\mathbf{x} - \mathbf{z}|^3} \times d\mathbf{z}, \quad (1)$$

where κ is the quantum of circulation and the Biot-Savart integral is suitably desingularized [8]. The numerical method to evolve configurations of vortex filaments has been described elsewhere [8,16]. Here it suffices to say that the filaments are discretized into a large, variable number of points N depending on the local curvature: points are removed in regions where filaments straighten and are added where the local radius of curvature becomes smaller. The time evolution is based on a 4th order Runge-Kutta scheme with variable time step Δt , which depends on the current minimum distance δ_{min} along points, which determines the frequency of rotation of the fastest kelvin wave perturbation. If two vortex strands become closer to each other than the local discretization along filaments, then, consistently with the orientation of the filaments, our numerical algorithm reconnects the strands [17], provided that the total length is decreased [18]. This condition [19] mimics the small kinetic energy losses at vortex reconnections discussed in Ref. [11]. We also checked that, with our algorithm, the reconnection condition used by Schwarz [8] is satisfied [20].

Using this model, we study the interaction of two bundles of a given number M of (initially) straight parallel vortex strands, set (initially) at 90° to each other. Let D be the distance between the axes of the two bundles and A be the radius of each bundle. The calculation is performed in a cubic periodic box $-B \leq x, y, z \leq B$. A typical result for $M = 7$, $A = 0.0417$ cm, $D = 3A$, and $B = 1$ cm is shown in Fig. 1. The initial position of vortex strands within the same bundle is symmetric: in this case we place six vortices at the corner of a hexagon and one vortex in the

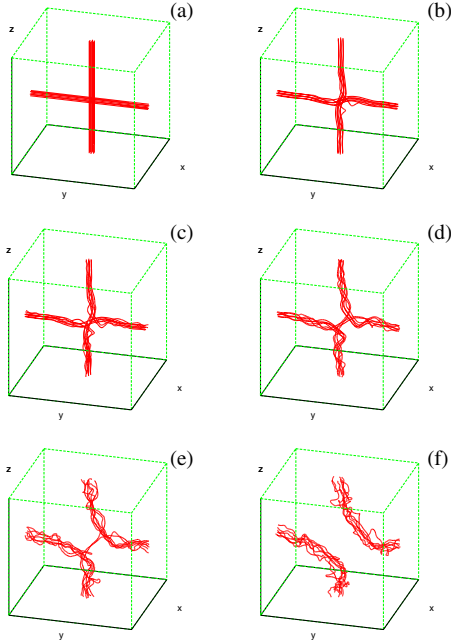


FIG. 1 (color online). Reconnection of two bundles of seven vortex strands each. (a) $t = 0$ s, (b) $t = 7.13$ s, (c) $t = 23.58$ s, (d) $t = 36.27$ s, (e) $t = 61.49$ s, (f) $t = 80.35$ s.

middle. In the absence of the second bundle, the vortex strands of the first bundle would rotate around each other in a time scale τ of the order of 10 s because they are parallel. With two bundles, the interaction makes them to bend in the direction of each other, until, at time $t_1 \approx 6.7$ s, the first reconnection occurs. The successive evolution involves the reconnections of all strands at times $t_2 \approx 16.4$ s, $t_3 \approx 21.0$ s, $t_4 \approx 23.6$ s, $t_5 \approx 27.4$ s, $t_6 \approx 52.0$ s, and $t_7 \approx 65.9$ s, after which the two bundles separate from each other (Fig. 1, bottom right) and move away.

Similar calculations with different parameters show that the last reconnection (at which the remaining vortex bridge breaks, and the bundles become free from each other) usually takes longer than the first few reconnections. Calculations also show that, after the first reconnection, the vortex strands develop kelvin waves of large amplitude. During the process, the total vortex length L increases by about 40%, as shown in Fig. 2. Figure 3 shows the average inverse radius of curvature, $\langle 1/R \rangle$, obtained by computing $|d^2\mathbf{s}/d\xi^2|$ at each point \mathbf{s}_j ($j = 1, \dots, N$) and then averaging over all points. To cope with the increase in L and decrease of $\langle R \rangle$, the number of points (initially $N = 700$) grows with time, up to $N = 4191$, when we stop this particular calculation. Figure 4 shows normalized histograms (probability density functions) of the average inverse curvature computed at different times: the formation of the kelvin wave cascade is evident in the development of a tail of the distribution at larger and larger times. All these results were confirmed by further calculations with $M = 5$ and 11 strands. If the number of strands of the first bundle

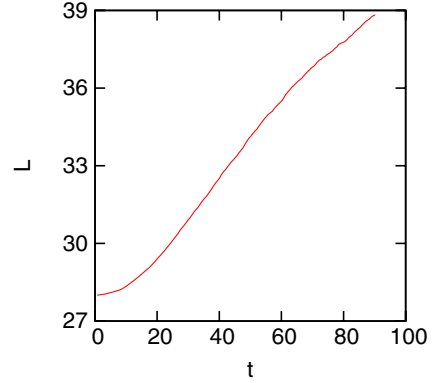


FIG. 2 (color online). Total vortex length L versus time t corresponding to Fig. 1.

is different from the number of the second bundle, the two bundles reconnect and move away as recognizable structures, although they are not fully separated as some vortex strands remain uncompensated. The uncompensated lines, deformed by the combined velocity field of the two bundles, may undergo further reconnections later, creating randomness in between the bundles, as visible in Ref. [15].

To make sure that our result does not depend on the reconnection algorithm, we study the interaction of vortex bundles by solving the NLSE, also called the Gross-Pitaevskii equation:

$$2i \frac{\partial \psi}{\partial t} = -\nabla^2 \psi + |\psi|^2 \psi - \psi. \quad (2)$$

The NLSE is a convenient model of superfluid He II [21]; for example, Koplik and Levine [9] used it to confirm Schwarz's insight that quantized vortices reconnect. Equation (2) is written in terms of $\hbar/\sqrt{2mE}$ (the coherence length) as unit of space, $2E/\hbar$ as unit of time, and mE/V as unit of density, where m is the mass of one boson, V the strength of the interaction between bosons, and E the energy per boson. The calculation is done using a 5th order Runge-Kutta-Fehlberg method (with typical time step of

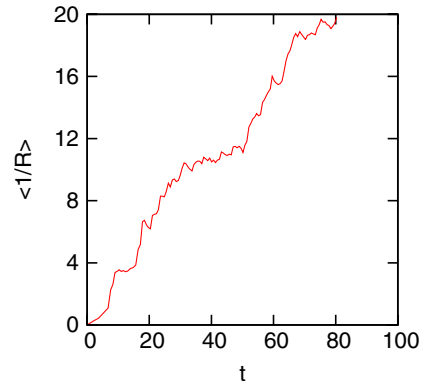


FIG. 3 (color online). Average inverse radius of curvature $\langle 1/R \rangle$ versus time t corresponding to Fig. 1.

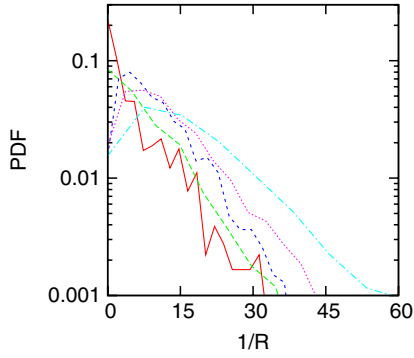


FIG. 4 (color online). Probability density function of the inverse radius of curvature at different times, corresponding to Fig. 1. From left to right, the times are $t = 7.13$ s (solid red curve), $t = 23.58$ s (long-dashed green curve), $t = 36.27$ s (dashed blue curve), $t = 61.49$ s (dotted purple curve), $t = 80.35$ s (dash-dotted light blue curve)

the order of 0.01) using 256^3 grid points and reflective boundary conditions.

In a second set of calculations, we thus solve the NLSE in a cubic box $-128 \leq x, y, z \leq 128$. The initial condition consists of two bundles of radius $A = 8$ and distance $D = 3A$ between the axes of the bundles. Each bundle contains $M = 7$ vortices (six at the corners of a hexagon and one in the middle). The time sequence, shown in Fig. 5, confirms the previous result that vortex bundles maintain their identity and reconnect with each other. Note the emission of small vortex rings in the last image. As in the previous calculation, large amplitude kelvin waves are generated. Figure 6 shows that at first L increases with t , then, when the bundles have moved sufficiently away from each other, L saturates; the relative increase of L is about 30%, confirming the intensification of vorticity. The relation between energy and length is important; it is common in the literature to state that the energy per unit length of vortex line is $[\rho_s \kappa^2 / (4\pi)] \ln(b/a)$ where ρ_s is the superfluid density and b is an upper cutoff (the radius of the container or the distance to the next vortex), but this relation is valid only for a straight vortex of course. The NLSE model allows one to determine the energy more precisely than the vortex filament model: in our calculation the total mass and the total energy are conserved within 1 part in 10^3 and 1 part in 10^4 , respectively. Further calculations with $M = 5$ and 9 strands confirm our results.

The above results refer to temperature $T = 0$. If $T > 0$ the motion of a vortex filament is governed by Schwarz's equation

$$\frac{ds}{dt} = \mathbf{v}_{si} + \alpha \mathbf{s}' \times (\mathbf{v}_n - \mathbf{v}_{si}) - \alpha' \mathbf{s}' \times [\mathbf{s}' \times (\mathbf{v}_n - \mathbf{v}_{si})], \quad (3)$$

where $\mathbf{s}' = ds/d\xi$ is the unit tangent at \mathbf{s} , \mathbf{v}_n is the normal fluid velocity, and α and α' are known temperature dependent coefficients resulting from the mutual friction be-

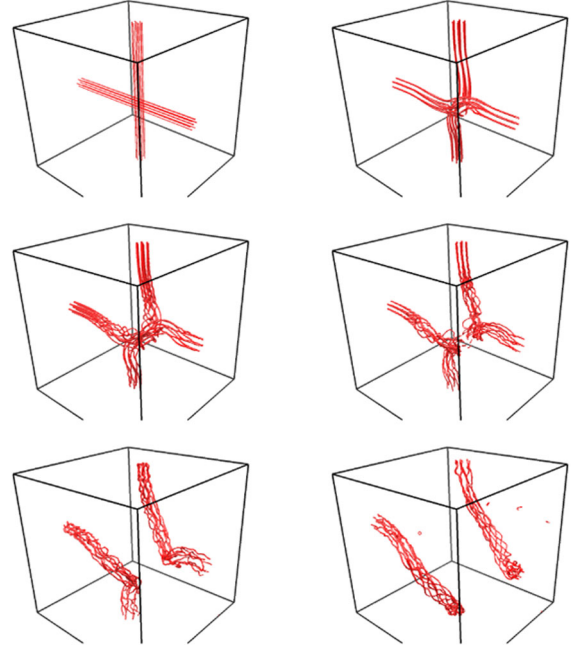


FIG. 5 (color online). Reconnection of two bundles of seven vortices each computed by solving the dimensionless NLSE. The figures show density isosurfaces at the level 0.25 (of the unit bulk density away from vortices) at different times t . The numerical resolution of the vortex core is such that at 0.25 density level there are about 3 grid points within a vortex core, and at 0.90 density level there are about 15 grid points. Top right: $t = 110$; middle left: $t = 240$; middle right: $t = 320$; bottom left: $t = 450$; bottom right: $t = 800$.

tween the vortex lines and the normal fluid [22]. In a third set of calculations we find that, at $T = 1.65$ K (the typical temperature of many experiments), with $\mathbf{v}_n = 0$, bundle reconnections are still possible, although with much less helical disturbances, as shown in Fig. 7, in agreement with calculations [17] which show that at $T > 0$ the vortex tangle is much smoother than at $T = 0$. We do not repeat this calculation with the NLSE model because there is not yet a consensus on how to generalize the NLSE to finite

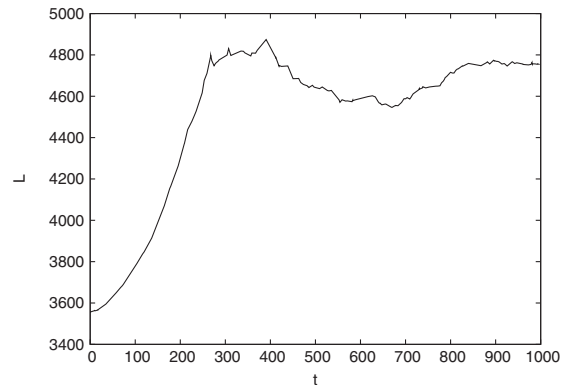


FIG. 6. Dimensionless vortex length L versus dimensionless time t corresponding to Fig. 5.

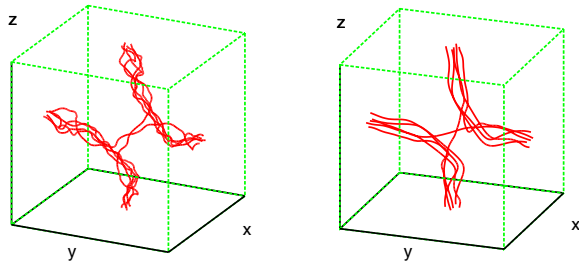


FIG. 7 (color online). Comparison between bundle reconnections at $T = 0$ K (left, corresponding to $\alpha = \alpha' = 0$) and at $T = 1.65$ K (right, corresponding to $\alpha = 0.111$ and $\alpha' = 0.01437$). The initial condition is the same for both calculations. The times (left, $t = 86.9$ s; right, $t = 58.1$ s) are chosen so that in each case the bundle reconnection has proceeded to the point that only two vortex strands are still part of the initial bundle.

temperatures, although many approaches have been proposed [23].

We conclude that vortex bundles are structurally stable structures, in the sense that they survive a time longer than their characteristic time τ , and travel a distance larger than their size A , consistently with results for bundles of vortex rings [24]. Remarkably, vortex bundles survive reconnections with other bundles without disintegrating, but rather amplifying their vortex length. Finally, the coiling of the vortex strands which is triggered by bundle reconnections confirms results of Holm and Kerr [25] about the generation of helicity in nearly singular vortex interactions of the Euler equation.

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[1] U. Frisch, *Turbulence* (Cambridge University Press, Cambridge, England, 1995).
 [2] S. Kida and M. Takaoka, *Annu. Rev. Fluid Mech.* **26**, 169 (1994).
 [3] R. J. Donnelly, *Quantised Vortices In Helium II* (Cambridge University Press, Cambridge, England, 1991).
 [4] W. F. Vinen and J. J. Niemela, *J. Low Temp. Phys.* **128**, 167 (2002); **129**, 213 (2002); C. F. Barenghi, *Physica (Amsterdam)* **237D**, 2195 (2008).
 [5] C. Nore, M. Abid, and M. E. Brachet, *Phys. Rev. Lett.* **78**, 3896 (1997); T. Araki, M. Tsubota, and S. K. Nemirowskii, *Phys. Rev. Lett.* **89**, 145301 (2002); M. Kobayashi and M. Tsubota, *Phys. Rev. Lett.* **94**, 065302 (2005); P. E. Roche *et al.*, *Europhys. Lett.* **77**, 66002 (2007); D. I. Bradley, S. N. Fisher, A. M. Guénault, R. P. Haley, S. O'Sullivan, G. R. Pickett, and V. Tsepelin (to be published).

[6] S. R. Stalp, L. Skrbek, and R. J. Donnelly, *Phys. Rev. Lett.* **82**, 4831 (1999); C. F. Barenghi, A. V. Gordeev, and L. Skrbek, *Phys. Rev. E* **74**, 026309 (2006); D. I. Bradley *et al.*, *Phys. Rev. Lett.* **96**, 035301 (2006); P. M. Walmsley *et al.*, *Phys. Rev. Lett.* **99**, 265302 (2007).
 [7] C. F. Barenghi and D. C. Samuels, *J. Low Temp. Phys.* **136**, 281 (2004).
 [8] K. W. Schwarz, *Phys. Rev. B* **38**, 2398 (1988).
 [9] J. Koplik and H. Levine, *Phys. Rev. Lett.* **71**, 1375 (1993).
 [10] W. F. Vinen, *Phys. Rev. B* **64**, 134520 (2001); C. F. Barenghi *et al.*, *J. Low Temp. Phys.* **138**, 629 (2005).
 [11] M. Leadbeater, T. Winiacki, D. C. Samuels, C. F. Barenghi, and C. S. Adams, *Phys. Rev. Lett.* **86**, 1410 (2001).
 [12] D. Kivotides, J. C. Vassilicos, D. C. Samuels, and C. F. Barenghi, *Phys. Rev. Lett.* **86**, 3080 (2001); W. F. Vinen, M. Tsubota, and A. Mitani, *Phys. Rev. Lett.* **91**, 135301 (2003); V. S. L'vov, S. V. Nazarenko, and G. E. Volovik, *JETP Lett.* **80**, 479 (2004); E. Kozik and B. Svistunov, *Phys. Rev. B* **77**, 060502(R) (2008).
 [13] G. P. Bewley, M. S. Paoletti, K. R. Sreenivasan, and D. P. Lathrop, arXiv:0801.2872.
 [14] I. Procaccia and K. R. Sreenivasan, *Physica (Amsterdam)* **237D**, 2167 (2008).
 [15] D. Kivotides, *Phys. Rev. Lett.* **96**, 175301 (2006); K. Morris, J. Koplik, and D. W. I. Rouson, *Phys. Rev. Lett.* **101**, 015301 (2008).
 [16] C. F. Barenghi *et al.*, *Phys. Fluids* **9**, 2631 (1997).
 [17] M. Tsubota, T. Araki, and S. K. Nemirowskii, *Phys. Rev. B* **62**, 11751 (2000).
 [18] D. C. Samuels, in *Quantized Vortex Dynamics and Superfluid Turbulence*, edited by C. F. Barenghi, R. J. Donnelly, and W. F. Vinen (Springer, New York, 2001), p. 97.
 [19] The identification of length and energy is correct under the local induction approximation [8], but is only an approximation under the Biot-Savart law.
 [20] Slightly different reconnection algorithms have been proposed, as in L. Kondaurova and S. K. Nemirowskii, *J. Low Temp. Phys.* **138**, 555 (2005).
 [21] P. H. Roberts and N. G. Berloff, in *Quantized Vortex Dynamics and Superfluid Turbulence* (Ref. [18]), p. 235.
 [22] C. F. Barenghi, R. J. Donnelly, and W. F. Vinen, *J. Low Temp. Phys.* **52**, 189 (1983).
 [23] S. K. Nemirowskii, *Theor. Math. Phys.* **141**, 1452 (2004); E. J. M. Madarassy and C. F. Barenghi, *J. Low Temp. Phys.* **152**, 122 (2008); B. Jackson, N. P. Proukakis, C. F. Barenghi, and E. Zaremba, arXiv:0809.4400; N. P. Proukakis, arXiv:0706.3541.
 [24] S. Alamri and C. F. Barenghi (to be published).
 [25] D. D. Holm and R. Kerr *Phys. Rev. Lett.* **88**, 244501 (2002); R. M. Kerr, *Nonlinearity* **9**, 271 (1996).