Cyclotron Maser Radiation from an Inhomogeneous Plasma

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Cyclotron maser radiation is important in both laboratory devices such as gyrotrons and in space physics applications to phenomena such as auroral kilometric radiation. To understand the behavior, especially in the latter case where there is generally a localized region of instability, requires an understanding of how such instabilities behave in an inhomogeneous plasma. Here we consider, for simplicity, a simple ring distribution of electrons in either a step function variation of magnetic field or a continuous gradient. In each case we show that there can exist localized regions of instability from which waves, growing in time, can be radiated outwards.

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Cyclotron maser instabilities are of importance in various astrophysical contexts such as planetary radio emission, for example, auroral kilometric radiation (AKR) [1–3], solar decimetric radiation [4], and in recently discovered emissions from rotating stars including periodic emission [5–7], astrophysical shocks [8], and blazars [9]. It has been argued in a number of previous papers that AKR and similar phenomena are likely to be produced by a distribution with a horseshoe or crescent shape in velocity space [10–12] and we have collaborated in setting up an experiment in which an electron beam moving along converging magnetic field lines produces a horseshoe distribution [13,14]. This experiment has produced radiation for which the spectral properties and the conversion efficiency from beam energy to radiation match those observed in AKR. In this paper we look at some effects of plasma nonuniformity on cyclotron maser instabilities. One long-standing problem in studies of cyclotron masers is how radiation, generally observed to be generated below the local cyclotron frequency, gets onto the branch of the dispersion relation which connects to vacuum propagation. It evidently does, since the AKR radiation is observed by satellites above the wave generation region, although the cold plasma dispersion curve goes to a cutoff at or above the cyclotron frequency and does not connect to the branch below the cyclotron frequency [15]. We shall show that the topology of the dispersion curves may be different in a plasma with an electron distribution showing a cyclotron maser instability and look at some properties of localized instabilities in an inhomogeneous plasma. Because of the complexity of the dispersion relation for a horseshoe distribution [10,16] we consider a system with a simple ring distribution of the form $f(v_\parallel, v_\perp) = n_0\delta(v_\parallel)\delta(v_\perp - v_0)$. Such a distribution has been considered in the literature on gyrotrons [17,18] (and references therein), and also in the space physics literature [8]. While it might be a reasonable representation of reality in the gyrotron, a monoenergetic distribution of this sort is perhaps less likely in the space plasma context. It is, however, simpler to treat since it gives rise to a simple algebraic dispersion relation. We shall show that in a nonuniform plasma it has some interesting and perhaps surprising properties which may cast some light on how more complex cyclotron maser instabilities might behave.

The dispersion relation for the ring distribution can readily be obtained by sustituting the ring distribution into the general dispersion relation as given, for example, by Stix [15]. Since relativistic effects play a crucial role in cyclotron maser instabilities the relativistic dielectric tensor elements must be used. We also make the approximation that the wave frequency is around the cyclotron frequency. The result is, for propagation parallel to the ambient magnetic field,

$$(\omega^2 - k^2c^2)\left[(\omega - \Omega)^2 + \frac{\omega_{pe}^2 T}{2\gamma}\right] - \frac{\omega_{pe}^2}{\gamma}\omega(\omega - \Omega) = 0, \tag{1}$$

and for perpendicular propagation

$$k^2c^2 = \omega^2\left[\varepsilon_\parallel - \frac{\varepsilon_\perp^2}{\varepsilon_\parallel}\right], \tag{2}$$

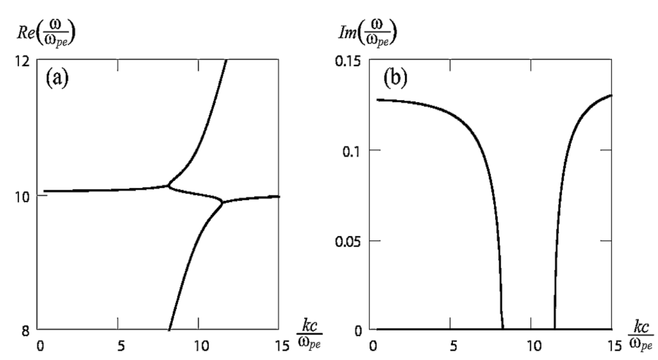

FIG. 1. The dispersion relation for parallel propagation for a ring distribution with $\gamma = 1.02$ and $\frac{\omega_{ce}}{\gamma\omega_{pe}} = 10$.

with

$$\varepsilon_{\parallel} = 1 - \frac{1}{2\gamma}\frac{\omega_{pe}^2}{\omega(\omega - \Omega)} + \frac{T}{4\gamma^2}\frac{\omega_{pe}^2}{(\omega - \Omega)^2},$$

$$\varepsilon_{\perp} = 1 - \varepsilon_{\parallel}, \quad \gamma = \left(1 - \frac{v_0^2}{c^2}\right)^{1/2}, \quad T = \frac{v_0^2}{c^2}, \quad \Omega = \frac{\omega_{ce}}{\gamma}.$$

Dispersion curves for the first case are shown in Figs. 1 and 2. The same qualitative behavior is seen for perpendicular propagation, with a purely real branch connecting the two unstable branches. This branch is not shown in the paper of Chu and Hirschfield [17] or in the later review by Chu [18] although it does appear in the paper by Pritchett [2]. It is clear that it must appear since the complex roots of the dispersion relation, which occur in conjugate pairs, must connect to pairs of real roots.

To begin our discussion of nonuniform plasmas we consider the case in which there is a region $0 < x < L$ in which the wave frequency is close to the cyclotron frequency and on each side regions in which it is well away from the cyclotron frequency, the variation of the dispersion properties with density being less important in the regime we are looking at. This is most realistically realized for perpendicular propagation where we can think of having discontinuities in the magnetic field strength along the direction of propagation. Now we note that away from the cyclotron resonance, the dispersion curves in a low density plasma are very close to those for a vacuum. For simplicity we suppose, therefore, that the dispersion relation outside the interval $[0, L]$ is just $\omega = kc$. Our objective is to investigate whether solutions can be found in which the mode grows in time and on each side of the slab there is an outward propagating wave. Such a solution represents a locally unstable layer radiating energy away on each side. So, we look for solutions of the form

$$\phi(x) = \left[\begin{array}{ll} e^{-ik_0x} & x < 0 \\ Ae^{ikx} + Be^{-ikx} & 0 < x < L \\ Ce^{ik_o(x-L)} & x > L \end{array}\right.,$$

where we can choose the amplitude of the wave in $x < 0$ to be unity (and there is, of course, a common factor e^{-iwt}). The wave number k in the central region is obtained from the dispersion relation for the plasma there, $F(\omega, k) = 0$ say, while k_0 is the vacuum wave number.

Imposing conditions of continuity of the amplitude and its derivative at the boundaries gives

$$A + B = 1,$$
$$Ae^{ikL} + Be^{-ikL} = C,$$
$$kA - kB = -k_0$$
$$kAe^{ikL} - kBe^{-ikL} = k_0C.$$

If we put $K = k/k_0$ and $l = k_0L$, then from the above we obtain the relation

$$i\tan(Kl) = \frac{2K}{K^2 + 1}, \tag{3}$$

which determines K. Note that this is independent of the dispersion relation for waves in the slab. All that is needed is that there be propagating waves on each side of it, the fact that we have assumed them to be symmetrical being just for convenience rather than strictly necessary. Possible solutions representing a locally unstable region emitting waves in both directions are found by first finding a value of K which satisfies (3). The wave number in the slab is then $k = Kk_0$ and from the external dispersion relation $k_0c = \omega$ so that $k = \frac{K\omega}{c}$. We must then look for a nontrivial solution, if such exists, of

$$F\left(\omega, \frac{K\omega}{c}\right) = 0. \tag{4}$$

In particular, we look for solutions with positive imaginary part, whose existence shows that there is an unstable layer which can emit growing waves in both directions. Equation (3) has multiple solutions, the first few of which are plotted in Fig. 2(a) for $L = 20$. A question of interest is just how the energy going off from the slab is partitioned between the two directions. From the above equations it can be shown that $C = \cos(Kl)[\frac{K^2-1}{K^2+1}]$ and that $|C^2| = 1$, showing that half the energy emitted from the slab goes in each direction, something which might have been expected from symmetry. In Figs. 2(b) and 2(c) we show some of the unstable modes predicted by (4) for a slab with $L = 20$ and

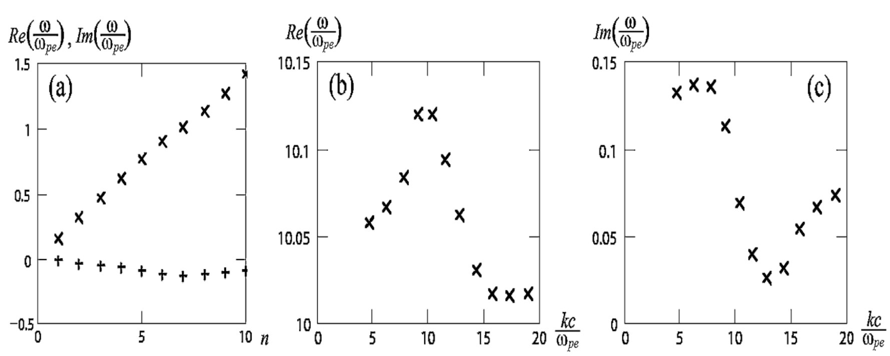

FIG. 2. (a) Solutions of Eq. (4) "×" real part, "+" imaginary part; (b) Real part of ω as a function of wave number; (c) Corresponding imaginary part.

the same parameters as were used in our plots of the infinite plasma dispersion relation. Since there is a discrete series of possible values of K, the dispersion curve now consists of a series of points. It is clear that the real part of the frequency of these unstable modes follows closely that of the infinite plasma, going along the connecting branch described above. The imaginary part gets smoothed out, so that the connecting branch is no longer stable although the growth rates are greater on each side of it. In these regions, where the infinite plasma is unstable, the finite slab growth rate is slightly below that for the infinite plasma. The existence of many eigenmodes with almost the same frequency means that modes with almost the same wavelength outside the slab can have very different wavelengths within the slab. This is reminiscent of the behavior we found for the horseshoe distribution when we analyzed the instability in a plasma annulus [16]. There a large number of modes were found with different structure within the plasma annulus but similar spatial growth rate.

Now we consider radiation from a nonuniform system in which the magnetic field varies so that

$$\Omega = \frac{eB}{\gamma m} = \Omega_0\left(1 + \frac{x}{L}\right), \tag{5}$$

i.e., the magnetic field has a gradient with characteristic scale length L. We scale all frequencies to be in terms of the plasma frequency and lengths in terms of c/ω_{pe}. The dispersion relation for both parallel and perpendicular propagation can be written in the form $k^2 = F(\omega, \Omega)$, which on inserting (5) can be converted into an equivalent differential equation

$$\frac{d^2\phi}{dx^2} = -F(\omega, \Omega(x))\phi = G(x)\phi. \tag{6}$$

We look at the properties of this equation. First we consider a simple WKB approximation. If we take $\omega = \Omega_0 = 10$, $\gamma = 1.02$ and $L = 20$ and parallel propagation, then the local wave number as a function of x is as in Fig. 3(a). As can be seen there is a smooth variation and we would not expect any reflection, a conclusion which can be confirmed by numerical solution of the equation. However, if we add a positive imaginary part to ω, corresponding to a growing wave, then by adjusting its magnitude we can obtain the dispersion curve shown in Fig. 3(b). With the correct choice of imaginary part, the real parts of the incident and reflected branches join up on the axis and the result is that a wave going to the left on the low field side connects up to a wave going to the right on the high field side. If we then solve the differential equation with the boundary condition of an outgoing wave on the low field side we obtain a solution of the form shown in Fig. 4. Away for the resonance the dispersion relation is close to being just $\omega^2 = k^2c^2$ and the imaginary part of ω produces a corresponding imaginary part of k so that the wave amplitude falls off exponentially in the direction of propagation. This is not damping, just a reflection of the fact that the further the wave is from the source the earlier the time at which it was produced and the lower its amplitude. The solution shown evidently corresponds to a solution with outgoing waves on both sides, an incoming component having an amplitude which increases exponentially away from the source. Such a wave is inevitably picked up at some point due to inaccuracies in the eigenvalue for ω and numerical error, but the solution shown is evidently close to an eigensolution with a localized instability and only outgoing waves. In fact the solution seems to correspond very closely to a situation in which there is only an outgoing wave on one side. For the parameters given and the real part chosen to be 10, the imaginary part of ω comes out at 0.13. The real part can be chosen arbitrarily and choosing it to be 10 simply means that the resonance $\mathrm{Re}(\omega) = \Omega$ is fixed to be at $x = 0$. This growth rate is of the same order of magnitude as the typical growth rate in a homogeneous plasma with the same parameters as those at $x = 0$ here. If we reduce the energy of the ring particles to make $\gamma = 1.01$, the growth rate is less at $\mathrm{Im}(\omega) = 0.09$, as

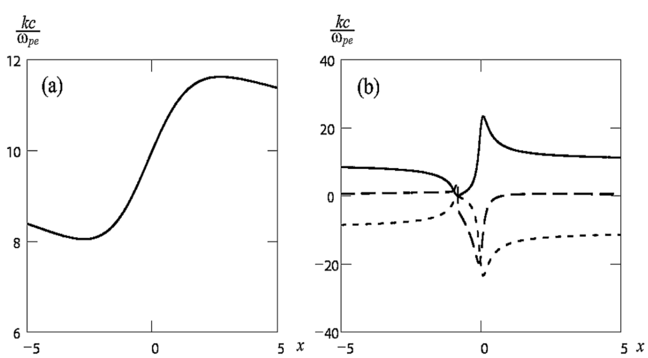

FIG. 3. (a) Wave number as a function of x with $\omega = 10$; (b) Wave number as a function of x with $\omega = 10 + 0.13.i$. In (b) the full and dotted lines show the real part, both positive and negative roots being plotted to illustrate the confluence of the roots. The dashed line is the imaginary part corresponding to the positive root.

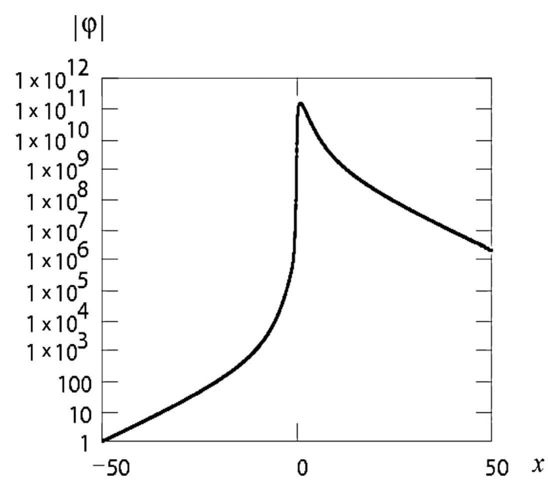

FIG. 4. Wave amplitude as a function of x.

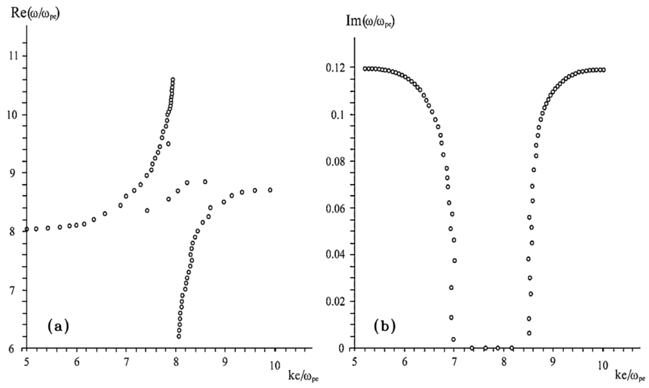

FIG. 5. Real (a) and imaginary (b) parts of ω for a horseshoe distribution.

might be expected. For perpendicular propagation we found similar behavior, but with the emission mainly to the low field side rather than the high field side as is the case for parallel propagation. The imaginary part of ω, 0.12, is again of the same order as in the corresponding homogeneous plasma and again lowering the γ of the ring lowers the growth rate, this time to around 0.08. In conclusion we see that for the continuous field gradient we can get, as in the slab geometry, solutions in which a localized region of instability gives rise to exponential growth of waves which propagate away from the resonance region. In both cases we have looked at, propagation exactly parallel and perpendicular to the magnetic field, the emission is on one side only. Why the direction of the emission changes in the different cases is something which needs further investigation. If the parameters are changed significantly from those at which the dispersion curves take the special form shown, then they seem generally to revert to something close to the real ω case in which there is almost perfect transmission of an incident wave.

We have considered the problem of a ring distribution, unstable to a cyclotron maser instability, in a plasma with a magnetic field gradient and have shown that it is possible to have a localized unstable region around the cyclotron resonance with waves radiating outwards from it. This distribution is unlikely to be a good approximation to the distribution in the auroral region but the more realistic horseshoe distribution leads, unfortunately, to a much more complicated dispersion relation solving which, even in the homogeneous plasma case, is not simple. However, Fig. 5 shows some solution values and that there is some similarity with the ring distribution. The main point to be taken from this analysis is that wave propagation in the presence of a distribution unstable to a cyclotron maser instability may be significantly different from that in a stable plasma. In an inhomogeneous system there can exist localized instabilities from which outwardly propagating waves are emitted, growing in time. Exactly how the emission behaves appears to depend on the geometry of the problem in ways which we have not yet managed to fully explain. Another important property is that nongrowing waves can propagate through the cyclotron resonance region, whereas in a cold plasma they would encounter a nonpropagating region with a cutoff on one side and a resonance on the other. While we cannot, with our simple ring distribution, claim to have solved the long-standing problem of how AKR gets onto the vacuum branch, what we have shown is that arguments based on the properties of the cold plasma dispersion curves do not necessarily indicate that there is a problem. To see how emission from a localized region of cyclotron instability behaves will need careful analysis of the wave properties with the relevant unstable distribution function. This Letter serves to show that interesting and perhaps unexpected results may emerge from such an analysis.

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