Evidence for Universal Scaling in the Spin-Glass Phase

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We perform Monte Carlo simulations of Ising spin-glass models in three and four dimensions, as well as of Migdal-Kadanoff spin glasses on a hierarchical lattice. Our results show strong evidence for universal scaling in the spin-glass phase in all three models. Not only does this allow for a clean way to compare results obtained from different coupling distributions, it also suggests that a so far elusive renormalization group approach within the spin-glass phase may actually be feasible.

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The characterization of the spin-glass phase in finite space dimensions remains as one of the prominent unresolved problems in the physics of disordered systems. Although the spectrum of theories that have been proposed to describe spin glasses is broad [1–9], the main theoretical pictures are the simple scaling approach (droplet picture) [2,3] and the replica symmetry breaking (RSB) scenario inspired by mean-field theory [1]. Because frustration and disorder are inherent ingredients of spin glasses, progress via field-theoretical calculations has been difficult, at least below six dimensions [10]. Thus most of the progress in the field relies on numerical studies that are also strongly limited mainly due to the numerical complexity of spin glasses. In fact, the difficulties are such that there is no good numerical evidence of whether a renormalization group (RG) approach may work in the spin-glass phase.

We address some of the necessary features that one should observe in the spin-glass phase such that scaling theory and an RG approach may be potentially successful. To check for the applicability of a scaling theory within the spin-glass phase we assume *a priori* that a scaling approach works and check *a posteriori* whether our results are consistent with the scaling assumptions made. For this purpose we study finite-size scaling functions where we compare the behavior of several observables on a change of scale: We rescale the system size and plot observables as a function of a phenomenological coupling which in our case is the Binder cumulant [11]. We find that this procedure leads to scaling functions consistent with a universal scaling behavior in the spin-glass phase for all models studied: the Migdal-Kadanoff (MK) spin glass on a (three-dimensional) hierarchical lattice [12], as well as the 3D and the 4D Edwards-Anderson (EA) short-range Ising spin glasses [13]. Of particular interest is the scaling function of the Binder cumulant itself since it provides a compact way to look at the complete RG flow.

The MK spin glass serves as an example to illustrate the behavior of the different finite-size scaling functions in a simple scaling theory. This, in turn, follows from the possibility to exactly solve the model (numerically) using an RG decimation transformation.

The physically interesting case of the 3D EA model is also the most difficult as the lower critical dimension is close [14], which means that the spin-glass phase in this case is rather marginal. Thus we also study the model in 4D. We observe similarities as well as clear differences between the MK and EA models. In particular, for all studied models we find clear evidence for the emergence of a universal scaling behavior in the spin-glass phase in the thermodynamic limit. Although our findings might be taken as a hint for the correctness of a simple scaling approach for the EA model, it is fair to observe that for very low temperatures we find an effective stiffness exponent θ which is compatible with zero, as expected in the RSB or TNT [7,8] scenarios. Furthermore, for the system sizes studied, the fractal dimension of the surface of the low-energy excitations seems not to be equal to the space dimension d, in agreement with the TNT or droplet scenarios. We therefore lay the foundation for a simple approach that should allow future studies to check whether the disagreement with the traditional pictures is due to scaling corrections, or whether new theoretical descriptions are needed.

Finite-size scaling approach.—For any observable $\mathcal{O}(L, T)$ as a function of the temperature T and the system size L, and the finite-size correlation length $\xi(L, T)$ finite-size scaling (FSS) theory predicts [15] that

$$\mathcal{O}(sL, T)/\mathcal{O}(L, T) = F_{\mathcal{O}}[\xi(L, T)/L; s] + (\text{corrections}),$$

where s is a scaling factor. The corrections vanish in the thermodynamic limit. Because even the correlation length divided by the system size diverges in the spin-glass phase, this definition is inconvenient to examine the behavior of FSS functions within the spin-glass phase. Hence we use an alternative phenomenological coupling as the scaling variable which allows for a better visualization of the scaling functions in the spin-glass phase. We find that the Binder cumulant $g(L, T)$ [see Eq. (2) below] works best since it is bounded in the interval [0,1], thus leading to a compact picture of the scaling properties in the whole spin-glass phase. In the following we study different FSS functions

$$\mathcal{O}(sL, T)/\mathcal{O}(L, T) = F_{\mathcal{O}}[g(L, T); s] + \text{(corrections)} \quad (1)$$

defined as a function of $g(L, T)$. Next, we verify numerically whether it is sensible to study such FSS functions in the spin-glass phase and discuss our findings.

Models and observables.—We consider the spin-glass Hamiltonian $\mathcal{H} = -\sum_{i,j} J_{ij}\sigma_i\sigma_j$, where the sum is over all nearest neighbor pairs on the lattice and $\sigma_i \in \{\pm 1\}$ are Ising spins. The couplings J_{ij} are drawn from either a Gaussian (G), bimodal (B), link-diluted bimodal (D), or irrational (I) distribution which corresponds to a bimodal distribution where half of the (randomly chosen) bonds are multiplied by an irrational constant $c_{\rm I} = (1 + \sqrt{5})/2$. The Hamiltonian is studied both on a hierarchical lattice with an effective space dimension of three and on a simple hypercubic lattice in three and four space dimensions. We use periodic boundary conditions for the EA model on the hypercubic lattices and free boundary conditions for the hierarchical lattice [16]. The space dimension of the hierarchical lattice with G generations is $d = 1 + \ln(b)/\ln(s)$, where b is the number of parallel branches and s is the number of bonds in series (we set $s = 2$ and $b = 4$ to obtain an effective space dimension of 3). The size of the system is $L = s^G$.

The spin-glass order parameter is given by $q = (1/N)\sum_i \sigma_i\tau_i$, where σ and τ are two replicas of the system with the same disorder. For the hierarchical lattice we use [17] $q = 1/(2N_L)\sum_{\langle ij\rangle}(\sigma_i\tau_i + \sigma_j\tau_j)$ where the sum runs over all links N_L of the lattice. The spin-glass susceptibility χ is defined via $\chi(L, T) = N[\langle q^2\rangle]_{\rm av}$, where $\langle\cdot\rangle$ represents a thermal average and $[\cdot]_{\rm av}$ a disorder average. The Binder cumulant g is defined as

$$g(L, T) = \frac{1}{2}\left(3 - \frac{[\langle q^4\rangle]_{\rm av}}{[\langle q^2\rangle]_{\rm av}^2}\right). \quad (2)$$

Finally, we also study the link-overlap $q_l = (1/N_L)\sum_{\langle ij\rangle}\sigma_i\sigma_j\tau_i\tau_j$, where the sum is over all links. Within the TNT picture the fractal dimension d_s of large-scale excitations can be extracted from the variance of the link overlap $\sigma_{ql}^2(L, T) = [\langle q_l^2\rangle - \langle q_l\rangle^2]_{\rm av} \sim L^{-\mu_l}$, where $\mu_l = \theta + 2(d - d_s)$ varies with temperature [18]. The stiffness exponent θ follows from the temperature derivative of the finite-size scaling function of the Binder cumulant $F_{\partial_T g}$ via the quotient method [19]:

$$s^{-\theta} = 1 + g^*\partial_g F_g(g, s)|_{g=g^*} + \text{(corrections)}, \quad (3)$$

where g^* is the value of the Binder cumulant at any given strong-coupling fixed-point [20] and s is the scale factor used in the definition of the scaling function. Note that cumulants of the order parameter as well as the correlation length ξ/L are RG invariant quantities often referred to as "phenomenological couplings." In contrast to traditional scaling analyses of the spin-glass phase which use *bare* (unrenormalized) couplings (e.g., temperature), here we use renormalized couplings (e.g., Binder ratio) as scaling variables, thus presenting a cleaner analysis.

Computational details.—For the hierarchical lattice we use a similar procedure to Ref. [21]. An alternative way to calculate the link- and spin-overlaps on the hierarchical lattice is given in Ref. [22]. For $L \le 16$ ($G \le 4$) we use $N_{\rm sa} = 5 \times 10^5$ samples, for $L = 32$ ($G = 5$) at least 10^5 samples, for $L = 64$ ($G = 6$) at least 9×10^4 samples and for $L = 128$ ($G = 7$) at least 1.5×10^4 samples. All data are averaged over 10^3 independent configurations.

For the regular lattices the simulations are performed using the exchange Monte Carlo method [23,24]. For systems with Gaussian disorder we use the equilibration test of Ref. [18], whereas for the link-diluted case we perform a logarithmic binning of the data. Once the last three bins agree within error bars the system is equilibrated. For the 3D model with irrational [Gaussian] couplings $T_c = 1.47(3)$ [$T_c = 0.951(9)$]. For the 4D link-diluted model with 35% dilution [26] [Gaussian disorder] $T_c = 1.0385(25)$ [$T_c = 1.805(10)$]. For details see Table I.

TABLE I. Parameters for the simulations of the 3D model with Gaussian (3DG) and irrational (3DI) disorder, as well as the 4D model with Gaussian (4DG) and bond-diluted (4DD) disorder. L is the system size, $N_{\rm sa}$ is the number of disorder realizations, $N_{\rm sw}$ is the number of equilibration and measurement sweeps, $T_{\rm min}$ is the lowest temperature and N_r the number temperatures used in the exchange Monte Carlo method.

Model	L	$N_{\rm sa}$	$N_{\rm sw}$	$T_{\rm min}$	N_r
3DG	4	109 212	1 048 576	0.20	22
	5	100 303	1 048 576	0.20	22
	6	101 643	1 048 576	0.20	22
	8	40 430	8 388 608	0.20	22
	10	10 687	3 3554 432	0.20	22
	12	5134	3 3554 432	0.42	18
	16	5003	8 388 608	0.50	17
3DI	4	160 000	256 000	0.30	17
	5	160 000	256 000	0.30	17
	6	160 000	512 000	0.30	17
	8	160 000	1 024 000	0.30	17
	10	23 712	4 096 000	0.30	26
	12	12 768	4 096 000	0.70	22
4DG	3	20 000	131 072	1.40	29
		100 000	16 384	0.39	20
	4	20 000	131 072	1.40	29
	5	20 000	131 072	1.40	29
	6	20 000	131 072	1.40	29
		10 025	4 194 304	0.39	20
	8	3500	524 288	1.40	29
	10	2000	524 288	1.40	29
4DD	3	11 392	100 000	0.50	11
		102 400	20 000	0.95	11
	4	107 680	40 000	0.95	11
	5	101 699	40 000	0.95	11
	6	3072	200 000	0.50	11
		101 664	40 000	0.95	11
	8	41 408	100 000	0.95	21
	10	24 160	100 000	0.95	21

Finite-size scaling functions.—We first address the scaling function of the Binder cumulant F_g for all models studied and then discuss further scaling functions. In Fig. 1 we show our results for the FSS function F_g for different models. F_g displays in a compact way the RG flow of the Binder cumulant. Panels (a) and (b) show results for the hierarchical model where the scaling approach is known to work. Panel (a) shows a comparison between Gaussian and irrational coupling distributions, whereas panel (b) shows how the FSS scaling function for the bimodal coupling distribution—amidst strong finite-size effects—converges (slowly) towards the FSS function of the Gaussian (and irrational) cases. The convergence is slowest close to $g = 1$ ($T = 0$) because for low T entropic effects become relevant [27]. These results are a clear indication for universal FSS *in* the spin-glass phase for the MK model. Panel (c) shows a comparison of the Gaussian and the irrational coupling distribution in the 3D EA model. Finally, panel (d) shows a comparison between the Gaussian and the link-diluted bimodal EA model in 4D. The results for the 3D and 4D EA models are consistent with the scaling hypothesis and indicate a universal scaling behavior. The broken lines indicates how the FSS function should depart from $g = 1$ assuming that simple droplet scaling is correct, i.e., that the slope is given by the exponent θ through Eq. (3). While this works perfectly for the MK model, there is a clear difference in the case of the EA model.

In order to show that the scaling collapse in Fig. 1 is not coincidental we show in Fig. 2 the FSS function of the spin-glass susceptibility. The top panel shows a comparison of Gaussian and irrational disorder for the MK model, whereas the bottom panel shows a comparison for the 3D EA model. Again, the data show strong evidence of a universal scaling behavior. The data for the MK model seem to fall onto a straight line which underlines the very simple scaling behavior in this model.

In Fig. 3 we show results for the link-overlap variance. The top panel shows data for the MK model with Gaussian and irrational disorder distributions. In contrast to the other FSS functions discussed so far we find sizable scaling cor-

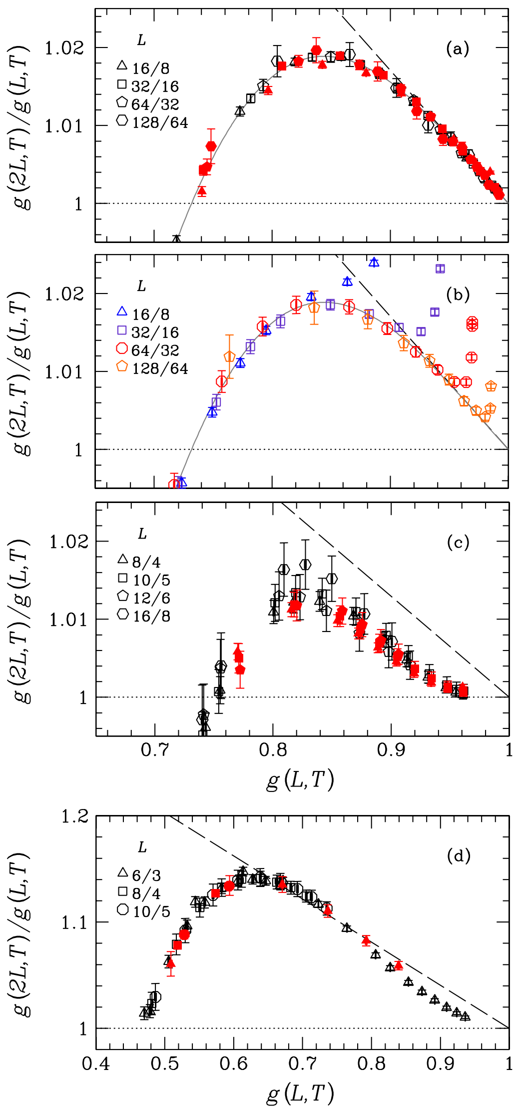

FIG. 1 (color online). Comparison of the finite-size scaling function of the Binder cumulant for different models. (a) Gaussian (open or black symbols) and irrational (closed or red symbols) disorder for the hierarchical lattice. (b) Hierarchical lattice with bimodal disorder. The data converge slowly to the Gaussian limiting case (solid curve). (c) 3D EA model with Gaussian (open or black symbols) and irrational (full or red symbols) disorder. (d) 4D EA model with both link-diluted (closed or red symbols) and Gaussian (open or black symbols) disorder. The dashed lines in all panels represent a droplet scaling behavior with $\theta = 0.27$ for the hierarchical, $\theta = 0.2$ for the 3D and $\theta = 0.75$ for the 4D models, respectively.

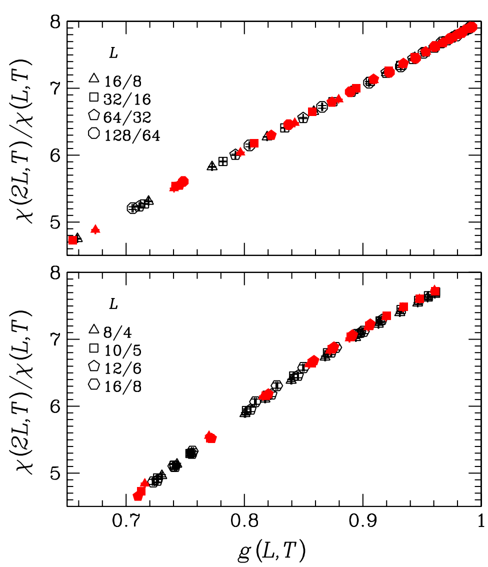

FIG. 2 (color online). Comparison of the finite-size scaling function of the susceptibility χ for the hierarchical lattice (top) with the 3D EA model (bottom). In both models the results for the Gaussian (open or black symbols) and the irrational (full or red symbols) coupling distribution are shown.

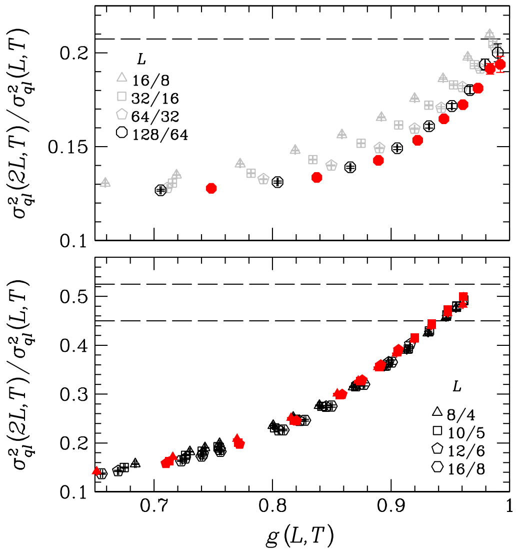

FIG. 3 (color online). Comparison of the finite-size scaling function of the link-overlap variance σ^2_{ql} for the hierarchical model (top) and the 3D EA model (bottom). For both MK an EA models the results for the Gaussian (open or black symbols) and the irrational disorder (full or red symbols) are shown. The broken line (the range between the broken lines, respectively) correspond(s) to the expected value of the scaling function for $g \rightarrow 1$ in the thermodynamic limit within the droplet model.

rections. The data are less conclusive but there is a clear trend that the curves for the two different distributions converge to a single master curve. The bottom panel shows data for the aforementioned disorder distributions in the 3D EA model. The data are similar to the MK case and show small corrections to scaling. The broken lines (or the region between them) correspond(s) to the expected value of the scaling function at $g = 1$ in the thermodynamic limit under droplet scaling assumptions (here we use $\theta = 0.20(5)$ and $d - d_s = 0.42(3)$ [8] in the 3D EA case; $\theta = 0.27$ and $d_s = d - 1 = 2$ in the MK case). Neither the droplet nor the RSB picture (where $F_{\sigma^2_{ql}} \rightarrow 1$) extrapolate to a consistent value for $F_{\sigma^2_{ql}}$ in the limit $g \rightarrow g(L, T = 0)$, in agreement with Refs. [7,8].

Summary and discussion.—Studying the behavior of several FSS functions we have found evidence for a universal scaling behavior *in* the spin-glass phase. The existence of these universal FSS functions allows us to perform for the first time a precise comparison of results in the spin-glass phase between different coupling distributions. We find that neither the simple droplet nor the RSB picture extrapolate to consistent values for the scaling functions in the zero-temperature limit. For both pictures it can be argued that these inconsistencies might be due to scaling corrections [28,29]. Our results suggest that in such a case not only the scaling behavior in the thermodynamic limit is universal, but also the leading scaling corrections. Although our results do not allow us to discriminate between the different scenarios that have been proposed for the nature of the spin-glass phase, they allow for a new, parameter-free way of looking at the problem.

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