Orthopositronium Lifetime: Analytic Results in $\mathcal{O}(\alpha)$ and $\mathcal{O}(\alpha^3 \ln\alpha)$

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(Received 30 June 2008; published 5 November 2008)

We present the $\mathcal{O}(\alpha)$ and $\mathcal{O}(\alpha^3 \ln\alpha)$ corrections to the total decay width of orthopositronium in closed analytic form, in terms of basic irrational numbers, which can be evaluated numerically to arbitrary precision.

DOI: 10.1103/PhysRevLett.101.193401 PACS numbers: 36.10.Dr, 12.20.Ds, 31.30.J−

Quantum electrodynamics (QED), the gauged quantum field theory of the electromagnetic interaction, celebrated ground-breaking successes in the twentieth century. In fact, its multiloop predictions for the anomalous magnetic moments of the electron and the muon were found to agree with highest-precision measurements within a few parts of 10^{-12} and 10^{-10}, respectively.

Another ultrapure laboratory for high-precision tests of QED is provided by positronium (Ps), the lightest known atom, being the electromagnetic bound state of the electron e^- and the positron e^+, which was discovered in the year 1951 [1]. In fact, thanks to the smallness of the electron mass m relative to typical hadronic mass scales, its theoretical description is not plagued by strong-interaction uncertainties and its properties, such as decay widths and energy levels, can be calculated perturbatively in nonrelativistic QED (NRQED) [2], as expansions in Sommerfeld's fine-structure constant α, with very high precision.

Ps comes in two ground states, $^1S_0$ parapositronium (p-Ps) and $^3S_1$ orthopositronium (o-Ps), which decay to two and three photons, respectively. In this Letter, we are concerned with the lifetime of o-Ps, which has been the subject of a vast number of theoretical and experimental investigations. Its first measurement [3] was performed later in the year 1951 and agreed well with its lowest-order (LO) prediction of 1949 [4]. Its first precision measurement [5], of 1968, had to wait nine years to be compared with the first complete one-loop calculation [6], which came two decades after the analogous calculation for p-Ps [7], which was considerably simpler owing to the two-body final state. In the year 1987, the Ann Arbor group [8] published a measurement that exceeded the best theoretical prediction available then by more than ten experimental standard deviations. This so-called o-Ps lifetime puzzle triggered an avalanche of both experimental and theoretical activities, which eventually resulted in what now appears to be the resolution of this puzzle. In fact, the 2003 measurements at Ann Arbor [9] and Tokyo [10],

$$\Gamma(\text{Ann Arbor}) = 7.0404(10\,\text{stat})(8\,\text{syst})\mu\text{s}^{-1},$$
$$\Gamma(\text{Tokyo}) = 7.0396(12\,\text{stat})(11\,\text{syst})\mu\text{s}^{-1}, \tag{1}$$

agree with each other and with the present theoretical prediction,

$$\Gamma(\text{theory}) = 7.039\,979(11)\ \mu\text{s}^{-1}. \tag{2}$$

The latter is evaluated from

$$\Gamma(\text{theory}) = \Gamma_0\left[1 + A\frac{\alpha}{\pi} + \frac{\alpha^2}{3}\ln\alpha + B\left(\frac{\alpha}{\pi}\right)^2 - \frac{3\alpha^3}{2\pi}\ln^2\alpha + C\frac{\alpha^3}{\pi}\ln\alpha\right], \tag{3}$$

where [4]

$$\Gamma_0 = \frac{2}{9}(\pi^2 - 9)\frac{m\alpha^6}{\pi} \tag{4}$$

is the LO result. The leading logarithmically enhanced $\mathcal{O}(\alpha^2 \ln\alpha)$ and $\mathcal{O}(\alpha^3\ln^2\alpha)$ terms were found in Refs. [11,12] and Ref. [13], respectively. The coefficients $A = -10.286\,606(10)$ [6,11,14–16], $B = 45.06(26)$ [15], and $C = -5.517\,024\,55(23)$ [17] are only available in numerical form so far. Comprehensive reviews of the present experimental and theoretical status of o-Ps may be found in Ref. [18].

Given the fundamental importance of Ps for atomic and particle physics, it is desirable to complete our knowledge of the QED prediction in Eq. (3). Since the theoretical uncertainty is presently dominated by the errors in the numerical evaluations of the coefficients A, B, and C, it is an urgent task to find them in analytical form, in terms of irrational numbers, which can be evaluated with arbitrary precision. In this Letter, this is achieved for A and C. The case of B is beyond the scope of presently available technology, since it involves two-loop five-point functions to be integrated over a three-body phase space. The quest for an analytic expression for A is a topic of old vintage: about 25 years ago, some of the simpler contributions to A, due to self-energy and outer and inner vertex corrections, were obtained analytically [19], but further progress then soon came to a grinding halt. The sustained endeavor of the community to improve the numerical accuracy of A [6,11,14–16] is now finally brought to a termination. An analytic expression for C is then simply obtained from that for A through the relationship [17]

$$C = \frac{A}{3} - \frac{229}{30} + 8\ln 2, \tag{5}$$

which may be understood qualitatively by observing that the $\mathcal{O}(\alpha^3 \ln\alpha)$ correction in Eq. (3) receives a contribution from the interference of the relativistic $\mathcal{O}(\alpha)$ term from hard scale with nonrelativistic $\mathcal{O}(\alpha^2 \ln\alpha)$ terms from softer scales.

The $\mathcal{O}(\alpha)$ contribution in Eq. (3), $\Gamma_1 = \Gamma_0 A\alpha/\pi$, is due to the Feynman diagrams where a virtual photon is attached in all possible ways to the tree-level diagrams, with three real photons linked to an open electron line, and the electron box diagrams with an e^+e^- annihilation vertex connected to one of the photons being virtual (see Fig. 1). Taking the interference with the tree-level diagrams, imposing e^+e^- threshold kinematics, and performing the loop and angular integrations, one obtains the two-dimensional integral representation [16]

$$\Gamma_1 = \frac{m\alpha^7}{36\pi^2}\int_0^1 \frac{dx_1}{x_1}\frac{dx_2}{x_2}\frac{dx_3}{x_3}\delta(2-x_1-x_2-x_3) \times [F(x_1,x_3) + \text{perm.}], \tag{6}$$

where x_i, with $0 \le x_i \le 1$, is the energy of photon i in the o-Ps rest frame normalized by its maximum value, the delta function ensures energy conservation, and *perm.* stands for the other five permutations of x_1, x_2, x_3. The function $F(x_1, x_3)$ is given by

$$F(x_1,x_3) = g_0(x_1,x_3) + \sum_{i=1}^{7} g_i(x_1,x_3)h_i(x_1,x_3), \tag{7}$$

where g_i are ratios of polynomials, which are listed in Eqs. (A5a)–(A5h) of Ref. [16], and

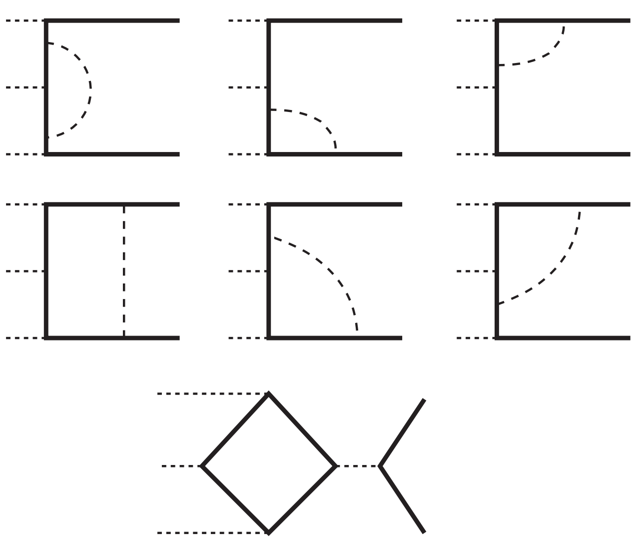

FIG. 1. Feynman diagrams contributing to the total decay width of o-Ps at $\mathcal{O}(\alpha)$. Self-energy diagrams are not shown. Dashed and solid lines represent photons and electrons, respectively.

$$\begin{aligned}
&h_1(x_1) = \ln(2x_1), \qquad h_2(x_1) = \sqrt{\frac{x_1}{\bar{x}_1}}\theta_1,\\
&h_3(x_1) = \frac{1}{2x_1}[\zeta_2 - \mathrm{Li}_2(1-2x_1)],\\
&h_4(x_1) = \frac{1}{4x_1}[3\zeta_2 - 2\theta_1^2], \qquad h_5(x_1) = \frac{1}{2\bar{x}_1}\theta_1^2,\\
&h_6(x_1,x_3) = \frac{1}{\sqrt{x_1\bar{x}_1x_3\bar{x}_3}}[\mathrm{Li}_2(r_A^+,\bar{\theta}_1) - \mathrm{Li}_2(r_A^-,\bar{\theta}_1)],\\
&h_7(x_1,x_3) = \frac{1}{2\sqrt{x_1\bar{x}_1x_3\bar{x}_3}}[2\mathrm{Li}_2(r_B^+,\theta_1) - 2\mathrm{Li}_2(r_B^-,\theta_1)\\
&\qquad - \mathrm{Li}_2(r_C^+,0) + \mathrm{Li}_2(r_C^-,0)],
\end{aligned} \tag{8}$$

with $\bar{x}_i = 1 - x_i$ and

$$\begin{aligned}
&\theta_1 = \arctan(\sqrt{\bar{x}_1/x_1}), \qquad \bar{\theta}_1 = \arctan(\sqrt{x_1/\bar{x}_1}),\\
&r_A^\pm = \sqrt{\bar{x}_1}\left(1 \pm \sqrt{\frac{x_1\bar{x}_3}{\bar{x}_1x_3}}\right), \qquad r_B^\pm = \sqrt{x_1}\left(1 \pm \sqrt{\frac{\bar{x}_1\bar{x}_3}{x_1x_3}}\right),\\
&r_C^\pm = r_B^\pm/\sqrt{x_1}.
\end{aligned} \tag{9}$$

Here, $\zeta_2 = \pi^2/6$ and

$$\mathrm{Li}_2(r,\theta) = -\frac{1}{2}\int_0^1 \frac{dt}{t}\ln(1 - 2rt\cos\theta + r^2t^2) \tag{10}$$

is the real part of the dilogarithm [see line below Eq. (21)] of complex argument $z = re^{i\theta}$ [20]. Since we are dealing here with a single-scale problem, Eq. (6) yields just one number.

Although Bose symmetry is manifest in Eq. (6), its evaluation is complicated by the fact that, for a given order of integration, individual permutations yield divergent integrals, which have to cancel in their combination. In order to avoid such a proliferation of terms, we introduce a regularization parameter, δ, in such a way that the symmetry unter $x_i \leftrightarrow x_j$ for any pair $i \neq j$ is retained. In this way, Eq. (6) collapses to

$$\Gamma_1 = \frac{m\alpha^7}{6\pi^2}\int_{2\delta}^{1-\delta} dx_1 \int_{1-x_1+\delta}^{1-\delta} \frac{dx_2}{x_1x_2x_3} F(x_1,x_3), \tag{11}$$

where $x_3 = 2 - x_1 - x_2$. Note that we may now exploit the freedom to choose any pair of variables x_i and x_j ($i \neq j$) as the arguments of F and as the integration variables.

The analytical integration of Eq. (11) is rather tedious and requires the use of some special techniques. For lack of space, we can only outline here a few examples. Specifically, we consider the last two functions of Eq. (8), which are most complicated. Using Eq. (10) and after some manipulations, we obtain the following integral representation for $h_7(x_1, x_3)$:

$$h_7(x_1,x_3) = -\frac{1}{4}\int_0^1 \frac{dt}{\sqrt{t}(x_1x_3-\bar{x}_1\bar{x}_3t)} \times\left[\ln\frac{\bar{x}_1x_3}{x_1\bar{x}_3} + 2\ln(x_3+\bar{x}_3t) - \ln t\right]. \quad (12)$$

Exploiting the $x_1 \leftrightarrow x_3$ symmetry of the coefficient $g_7(x_1,x_3)$ multiplying $h_7(x_1,x_3)$, this can be simplified as

$$h_7(x_1,x_3) = -\frac{1}{4}\int_0^1 \frac{dt}{\sqrt{t}(x_1x_3-\bar{x}_1\bar{x}_3t)}[2\ln(x_3+\bar{x}_3t) - \ln t]. \quad (13)$$

At this point, it is useful to change the order of integrations. Observing that the logarithmic terms in Eq. (13) are x_1 independent, we first integrate over x_1 (for a similar approach, see Ref. [21]). In order to avoid the appearance of complicated functions in the intermediate results, the integration over t in Eq. (13) is performed last.

Analogously, $h_6(x_1,x_3)$ can be rewritten as

$$h_6(x_1,x_3) = -\frac{1}{2}\int_0^1 \frac{dt}{\sqrt{t}(\bar{x}_1x_3-x_1\bar{x}_3t)}[\ln x_1 - \ln x_3 + \ln(x_3+\bar{x}_3t)], \quad (14)$$

in which the part proportional to $\ln x_1$ and the complementary part are first integrated over x_3 and x_1, respectively. The t integration is again performed last.

Let us now consider a typical integral that arises upon the first integration:

$$I = \int_0^1 \frac{dt}{t}\int_0^1 \frac{dx}{x}\ln[1-4t(1-t)(1-x)]\ln(1-x). \quad (15)$$

Direct integration over t or x would lead to rather complicated functions in the remaining variable. Instead, we Taylor expand the first logarithm using $\ln(1-x) = -\sum_{n=1}^{\infty} x^n/n$ to obtain

$$I = -\sum_{n=1}^{\infty}\frac{4^n}{n}\int_0^1\frac{dt}{t}[t(1-t)]^n\int_0^1\frac{dx}{x}(1-x)^n\ln(1-x). \quad (16)$$

Now the two integrals are separated and can be solved in terms of Euler's Gamma function, $\Gamma(x) = \int_0^\infty dt e^{-t}t^{x-1}$. Using

$$\int_0^1\frac{dx}{x}(1-x)^n\ln(1-x) = -\psi'(n+1), \quad (17)$$

where $\psi(x) = d\ln\Gamma(x)/dx$ is the digamma function, we finally have

$$I = \sum_{n=1}^{\infty}\frac{4^n}{2n}\frac{\Gamma^2(n)}{\Gamma(2n)}\psi'(n+1). \quad (18)$$

Another class of typical integrals yields sums involving digamma functions of half-integer arguments, e.g.

$$J = \int_0^1\frac{dt}{t}\int_0^1 dx\frac{\ln[1+4t(1-t)(1-x)]\ln(1-x)}{x-2} = \sum_{n=1}^{\infty}\frac{(-4)^n}{8n}\frac{\Gamma^2(n)}{\Gamma(2n)}\left[\psi'\left(\frac{n+2}{2}\right) - \psi'\left(\frac{n+1}{2}\right)\right]. \quad (19)$$

I and J belong to the class of so-called inverse central binomial sums [22,23], and methods for their summation are elaborated in Ref. [23]. With their help, I and J can be expressed in terms of known irrational constants, as

$$\begin{aligned} I &= -4\zeta_2 l_2^2 - \frac{l_2^4}{3} - 8\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{17}{2}\zeta_4, \\ J &= -\frac{3}{2}\zeta_2 l_2^2 + \frac{l_2^4}{4} - 3\zeta_2 l_2 l_r + l_2^2 l_r^2 + \frac{11}{12}l_2 l_r^3 \\ &\quad + \frac{47}{288}l_r^4 + 4l_2 l_r \mathrm{Li}_2(r) + \frac{7}{6}l_r^2\mathrm{Li}_2(r) - 6l_2\mathrm{Li}_3(-r) \\ &\quad - 2l_r\mathrm{Li}_3(-r) + 5l_2\mathrm{Li}_3(r) + \frac{4}{3}l_r\mathrm{Li}_3(r) + 6\mathrm{Li}_4\left(\frac{1}{2}\right) \\ &\quad + 4\mathrm{Li}_4(-r) - 5\mathrm{Li}_4(r) - \frac{13}{3}l_r S_{1,2}(r) + \frac{2}{3}S_{1,2}(r^2) \\ &\quad - 4S_{2,2}(-r) + 5S_{2,2}(r) + \zeta_3 l_2 + \frac{19}{6}\zeta_3 l_r, \end{aligned} \quad (20)$$

where $r = (\sqrt{2}-1)/(\sqrt{2}+1)$, $l_x = \ln x$,

$$S_{n,p}(x) = \frac{(-1)^{n+p-1}}{(n-1)!p!}\int_0^1\frac{dt}{t}\ln^{n-1}t\ln^p(1-tx) \quad (21)$$

is the generalized poly-logarithm, $\mathrm{Li}_n(x) = S_{n-1,1}(x)$ is the poly-logarithm of order n, and $\zeta_n = \zeta(n) = \mathrm{Li}_n(1)$, with $\zeta(x)$ being Riemann's zeta function [20,24].

Unfortunately, not all integrals can be computed so straightforwardly. In more complicated cases, the integrations are not separated after expansion into infinite series. We then end up with nested series that cannot be summed analytically with known algorithms, or just with one-dimensional integrals over complicated, but smooth functions. However, these can be evaluated numerically to a precision sufficiently high for an unambiguous application of the PSLQ algorithm [25], which allows one to reconstruct the representation of a numerical result known to very high precision in terms of a linear combination of a set of irrational constants with rational coefficients, if that set is known beforehand. In fact, the experience gained with the explicit solution of the simpler integrals already allows us to exhaust the relevant sets, so that no guesswork is necessary. In order for PSLQ to work reliably, which may be monitored through a specific confidence parameter, knowledge of the individual integrals up to typically 150 decimal figures is sufficient, while we generally achieve over 400 decimal figures. Once an analytical result is found, each additional decimal figure beyond the ones used as input in the first successful run of PSLQ provides a highly nontrivial check. The true analytical result can thus be safely established beyond any doubt.

After a laborious calculation, we obtain as the sum of several thousand individual integrals

$$\frac{2}{9}(\pi^2-9)A=\frac{56}{27}-\frac{901}{216}\zeta_2-\frac{11303}{192}\zeta_4+\frac{19}{6}l_2-\frac{2701}{108}\zeta_2 l_2+\frac{253}{24}\zeta_2 l_2^2+\frac{251}{144}l_2^4+\frac{913}{64}\zeta_2 l_3^2+\frac{83}{256}l_3^4-\frac{21}{4}\zeta_2 l_2 l_r$$
$$-\frac{49}{16}\zeta_2 l_r^2+\frac{7}{16}l_2 l_r^3+\frac{35}{384}l_r^4+\frac{581}{16}\zeta_2\mathrm{Li}_2\left(\frac{1}{3}\right)-\frac{21}{2}l_2\mathrm{Li}_3(-r)-\frac{7}{2}l_r\mathrm{Li}_3(-r)+\frac{63}{4}l_2\mathrm{Li}_3(r)+\frac{63}{8}l_r\mathrm{Li}_3(r)$$
$$-\frac{249}{32}\mathrm{Li}_4\left(-\frac{1}{3}\right)+\frac{249}{16}\mathrm{Li}_4\left(\frac{1}{3}\right)+\frac{251}{6}\mathrm{Li}_4\left(\frac{1}{2}\right)+7\mathrm{Li}_4(-r)-7S_{2,2}(-r)-\frac{63}{4}\mathrm{Li}_4(r)+\frac{63}{4}S_{2,2}(r)+\frac{11449}{432}\zeta_3$$
$$-\frac{91}{6}\zeta_3 l_2-\frac{35}{8}\zeta_3 l_r+\frac{1}{\sqrt{2}}\left[\frac{49}{2}\zeta_2 l_r-\frac{7}{72}l_r^3-\frac{35}{6}l_r\mathrm{Li}_2(r)+\frac{35}{6}\mathrm{Li}_3(r)-\frac{175}{3}S_{1,2}(r)+\frac{14}{3}S_{1,2}(r^2)+\frac{119}{3}\zeta_3\right]. \quad (22)$$

The terms proportional to $1/\sqrt{2}$, whose appearance is unexpected at first sight [19], originate from the summands labeled $i = 2, 6, 7$ in Eq. (7) and were all found by direct integration.

From Eqs. (22) and (5), A and C can be numerically evaluated with arbitrary precision,

$$A = -10.286\,614\,808\,628\,262\,240\,150\,169\,210\,991\ldots,$$
$$C = -5.517\,027\,491\,729\,858\,271\,378\,866\,098\,665\ldots. \quad (23)$$

These numbers agree with the best existing numerical evaluations [15,16] within the quoted errors.

In conclusion, we obtained the $\mathcal{O}(\alpha)$ and $\mathcal{O}(\alpha^3\ln\alpha)$ corrections to the total decay width of o-Ps, i.e., the coefficients A and C in Eq. (3), respectively, in closed analytic form. Another important result is the appearance of new irrational constants in Eq. (22). These constants enlarge the class of the known constants in single-scale problems. The constant B in Eq. (3) still remains analytically unknown.

We are grateful to G. S. Adkins for providing us with the computer code employed for the numerical analysis in Ref. [16]. This work was supported in part by BMBF Grant No. 05 HT6GUA, DFG Grant No. SFB 676, and HGF Grant No. NG-VH-008.

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