Colorful Horizons with Charge in Anti–de Sitter Space

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An Abelian gauge symmetry can be spontaneously broken near a black hole horizon in anti–de Sitter space using a condensate of non-Abelian gauge fields. A second order phase transition is shown to separate Reissner–Nordström–anti–de Sitter solutions from a family of symmetry-breaking solutions which preserve a diagonal combination of gauge invariance and spatial rotational invariance.

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Introduction.—Black holes and black branes usually prefer to be as symmetrical as possible. For example, the Schwarzschild black hole in four dimensions has spherical symmetry $SO(3)$, and it is stable against perturbations that break this symmetry [1].

There are exceptions to the rule that horizons prefer to be symmetrical, and they are associated with interesting physics. For example, the Gregory-Laflamme instability [2] of horizons with a translational symmetry is essentially a spinodal instability [3]. In [4], I studied ways in which black hole horizons could spontaneously break global or gauge symmetries of gravity coupled to an appropriate matter Lagrangian. But the matter Lagrangians in question were fairly complicated, with several parameters and, typically, nonrenormalizable interactions.

In [5,6], the search for simpler examples of spontaneous symmetry breaking near black hole horizons intersected with attempts [7–10] to find analogs in the gauge-string duality [11–13] of phenomena associated with superconductors. Coupling the Abelian Higgs model to gravity plus a negative cosmological constant leads to black hole solutions where a condensate of the complex scalar near the horizon spontaneously breaks the Abelian gauge symmetry. Insofar as such symmetry breaking signals superconductivity, such solutions can be regarded as superconducting black holes. The black hole metrics are asymptotically anti–de Sitter, and their dual descriptions involve an expectation value of a scalar operator which spontaneously breaks a global $U(1)$ symmetry. If this symmetry is weakly gauged, then superconducting black holes translate to superconducting states in a dual conformal field theory. If the symmetry is not gauged in the boundary theory, then one should think of the dual to a superconducting black hole as a form of superfluidity.

Unpublished numerical calculations based on the model proposed in [5] indicate that superconducting horizons are thermodynamically preferred below some nonzero critical temperature. Further evidence in this direction appeared in [6] in a strong coupling limit. But there are enough parameters in the Lagrangians considered in [4–6] that it is challenging to characterize the degree of universality or robustness of such numerical results. I will therefore consider an analogous phenomenon in a theory whose Lagrangian is mostly determined by symmetry principles:

$$I = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\,\mathcal{L}, \tag{1}$$

where, in mostly plus signature,

$$\mathcal{L} = R + \frac{6}{L^2} - \frac{1}{4}(F^a_{\mu\nu})^2, \tag{2}$$

and

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc}A^b_\mu A^c_\nu \tag{3}$$

is the field strength of an $SU(2)$ gauge field. Here ϵ^{abc} is the totally antisymmetric tensor with $\epsilon^{123} = 1$.

The gravitational coupling κ enters the action (1) only as a prefactor, so it cancels out of the equations of motion. Thus, the only dimensionless parameter in the equations of motion following from (1) is gL.

It is convenient to represent the gauge field as a matrix-valued one-form:

$$A = A^a_\mu \tau^a dx^\mu, \tag{4}$$

where $\tau^a = \sigma^a/2i$, so that

$$[\tau^a, \tau^b] = \epsilon^{abc}\tau^c. \tag{5}$$

(By σ^a I mean the usual Pauli matrices.) I will restrict attention to the following ansatz:

$$ds^2 = e^{2a}(-h dt^2 + dx_1^2 + dx_2^2) + \frac{dr^2}{e^{2a}h}, \tag{6a}$$

$$A = \Phi\tau^3 dt + w(\tau^1 dx_1 + \tau^2 dx_2), \tag{6b}$$

where a, h, Φ, and w are functions only of r. The gauge field (6b) is a slight simplification of the ansatz considered in [14,15]. A substantial literature has grown up around similar solutions of Einstein-Yang-Mills theory; see [16,17] for reviews.

The electrostatic potential Φ must vanish at the horizon for A to be well defined as a one-form, but I do not require it to vanish at infinity; thus the black hole can carry charge under the $U(1)$ gauge symmetry generated by τ^3.

The condensate $w(\tau^1 dx_1 + \tau^2 dx_2)$ breaks the $U(1)$ rotation symmetry in the x_1-x_2 plane as well as the $U(1)$ gauge symmetry generated by τ^3, but it preserves a combination of the two. A less symmetric ansatz could be considered, where the $\tau^1 dx_1$ and $\tau^2 dx_2$ components of A have different coefficient functions. This might result in a lower free energy, but it is significantly harder to study because the metric would probably not have $g_{11} = g_{22}$. In the interest of demonstrating spontaneous symmetry breaking in the cleanest possible finite-temperature setting, I will stick with the ansatz (6).

I will require w to be normalizable in the sense of making a finite contribution to the norm [18]

$$||A||^2 \equiv \int_{r_H}^{\infty} dr \sqrt{-g} g^{\mu\nu} A_\mu^a A_\nu^a, \tag{7}$$

where r_H is the location of the horizon. Normalizability of w is what it will mean for the condensate to form "near" the horizon. It is an appropriate requirement in the context of studying spontaneous symmetry breaking. Heuristically, solutions with normalizable w are the ones which can be reached, at least asymptotically, through real-time evolution starting from some solution with w close to 0—provided constraints like horizon area increase are satisfied.

In summary: The only conserved quantities associated with the black hole should be its mass density and its electric τ^3 charge density, and w, if it is nonzero, is a condensate whose presence spontaneously breaks $U(1)$ of spatial rotations times $U(1)$ of τ^3 into a diagonal subgroup. The aim of this Letter is to find out when this spontaneous symmetry breaking occurs.

Symmetry-breaking solutions.—Plugging (6a) and (6b) into the equations of motion following from (2) results in four second-order equations of motion:

$$a'' + a'^2 + \frac{1}{2} e^{-2a} w'^2 + \frac{g^2}{2} e^{-6a} \frac{\Phi^2 w^2}{h^2} = 0, \tag{8a}$$

$$h'' + 4a'h' - e^{-2a}\Phi'^2 - g^2 e^{-6a} \frac{\Phi^2 w^2}{h} + e^{-2a} h w'^2 - e^{-2a} h w'^2 - g^2 e^{6a} w^4 = 0, \tag{8b}$$

$$\Phi'' + 2a'\Phi' - 2g^2 e^{-4a} \frac{\Phi w^2}{h} = 0, \tag{8c}$$

$$w'' + \left(2a' + \frac{h'}{h}\right) w' + g^2 e^{-4a} \left(\frac{\Phi^2 w}{h^2} - \frac{w^3}{h}\right) = 0, \tag{8d}$$

together with a zero-energy constraint,

$$12 e^{2a} a'^2 + 4 e^{2a} \frac{h'}{h} a' - \frac{12}{L^2 h} + \frac{\Phi'^2}{h} - 2w'^2 - 2g^2 e^{-4a} \frac{\Phi^2 w^2}{h^2} + g^2 e^{-4a} \frac{w^4}{h} = 0. \tag{9}$$

The simplest solution of these equations with $\Phi \neq 0$ is the Reissner-Nordström black hole in anti–de Sitter space (hereafter RNAdS). The RNAdS solution has $w = 0$. Its explicit form can be found, for example, in [19]. The strategy pursued below is to numerically construct solutions with nonzero, normalizable w and compare their free energy with the RNAdS solution with the same temperature and charge density.

A solution to (8) and (9) with a regular horizon at $r = 0$ admits a near-horizon series expansion

$$a = a_0 + a_1 r + a_2 r^2 + \ldots \qquad h = h_1 r + h_2 r^2 + \ldots$$
$$\Phi = \Phi_1 r + \Phi_2 r^2 + \ldots \qquad w = w_0 + w_1 r + w_2 r^2 + \ldots. \tag{10}$$

Near the conformal boundary of anti–de Sitter space, with w normalizable, one finds the expansions

$$a = \log\frac{r}{L} + \alpha_0 + \ldots,$$
$$h = H_0 + H_3 e^{-3a} + H_4 e^{-4a} + \ldots, \tag{11}$$
$$\Phi = p_0 + p_1 e^{-a} + \ldots, \qquad w = W_1 e^{-a} + \ldots.$$

One may compute the energy density, entropy density, temperature, chemical potential, and charge density, as well as an order parameter J, to be discussed further below, as follows:

$$\epsilon = -\frac{H_3}{\kappa^2 L H_0}, \qquad s = \frac{2\pi}{\kappa^2} e^{2a_0}, \qquad \mu = \frac{p_0}{2L\sqrt{H_0}},$$
$$T = \frac{1}{4\pi} e^{2a_0} \frac{h_1}{\sqrt{H_0}}, \qquad \rho = -\frac{p_1}{\kappa^2 \sqrt{H_0}}, \qquad J = \frac{W_1 L}{\kappa^2}. \tag{12}$$

All dependence on κ can be removed by defining

$$\hat{X} = \frac{\kappa^2}{(2\pi)^3 L^2} X, \tag{13}$$

where X is an extensive thermodynamic density such as ϵ, s, ρ, J, or the long $f = \epsilon - Ts$. The factors of 2π are included for later convenience.

Any relation among thermodynamic quantities must be expressible in terms of dimensionless ratios. For example, the RNAdS solutions have

$$\frac{\hat{\epsilon}}{\hat{s}^{3/2}} = 1 + \frac{\pi^2 \hat{\rho}^2}{\hat{s}^2}. \tag{14}$$

Of special interest is the dimensionless ratio $\Delta\hat{f}/\hat{\rho}^{3/2}$, where Δf is the difference in the free energy density, between a symmetry-breaking solution and the RNAdS solution with the same T and ρ. $\Delta f < 0$ means that the symmetry-breaking solution is preferred.

When $w \neq 0$, one may ask what fraction q of the electric charge density is carried by the non-Abelian gauge bosons outside the horizon. The ratio of the flux of the τ^3 electric field through the horizon to the flux at infinity is $1 - q$. Therefore,

$$q = 1 + L e^{2a_0} \sqrt{H_0} \frac{\Phi_1}{p_1}. \tag{15}$$

In the dual field theory, $q = \rho_s/(\rho_s + \rho_n)$, where ρ_n and ρ_s are the charge densities in the normal and superfluid components, respectively.

The dual field theory has currents J_m^a satisfying an $SU(2)$ current algebra, where m runs over the t and x_i directions. The symmetry breaking that arises from nonzero w corresponds to expectation values

$$\langle J_i^a \rangle \propto J\delta_i^a, \tag{16}$$

where i runs over the values 1, 2. The tensor δ_i^a exhibits the locking of a spatial $U(1)$ and a gauge $U(1)$. (16) describes a form of long-range order which infrared fluctuations probably destroy; however, fluctuations are suppressed in the supergravity approximation, where $\kappa \ll L$ [20]. I will therefore ignore them.

Summary of results.—Here are the main results of my investigation of black hole solutions of the form (6).

(a) There is a two-parameter family of black hole solutions with positive, normalizable w. A convenient choice of parameters is gL and $T/\sqrt{\hat{\rho}}$.

(b) These superconducting black holes exist below a critical temperature T_c. $T_c/\sqrt{\hat{\rho}}$ is a function only of gL. T_c appears to go to zero at $gL \approx 0.8$. T_c is the temperature below which the RNAdS solutions exhibit an instability toward forming a condensate of the form (6b). This instability is reminiscent of the Nielsen-Olsen instability [21], but distinct because it is caused by color-electric fields and happens equally for charged scalars.

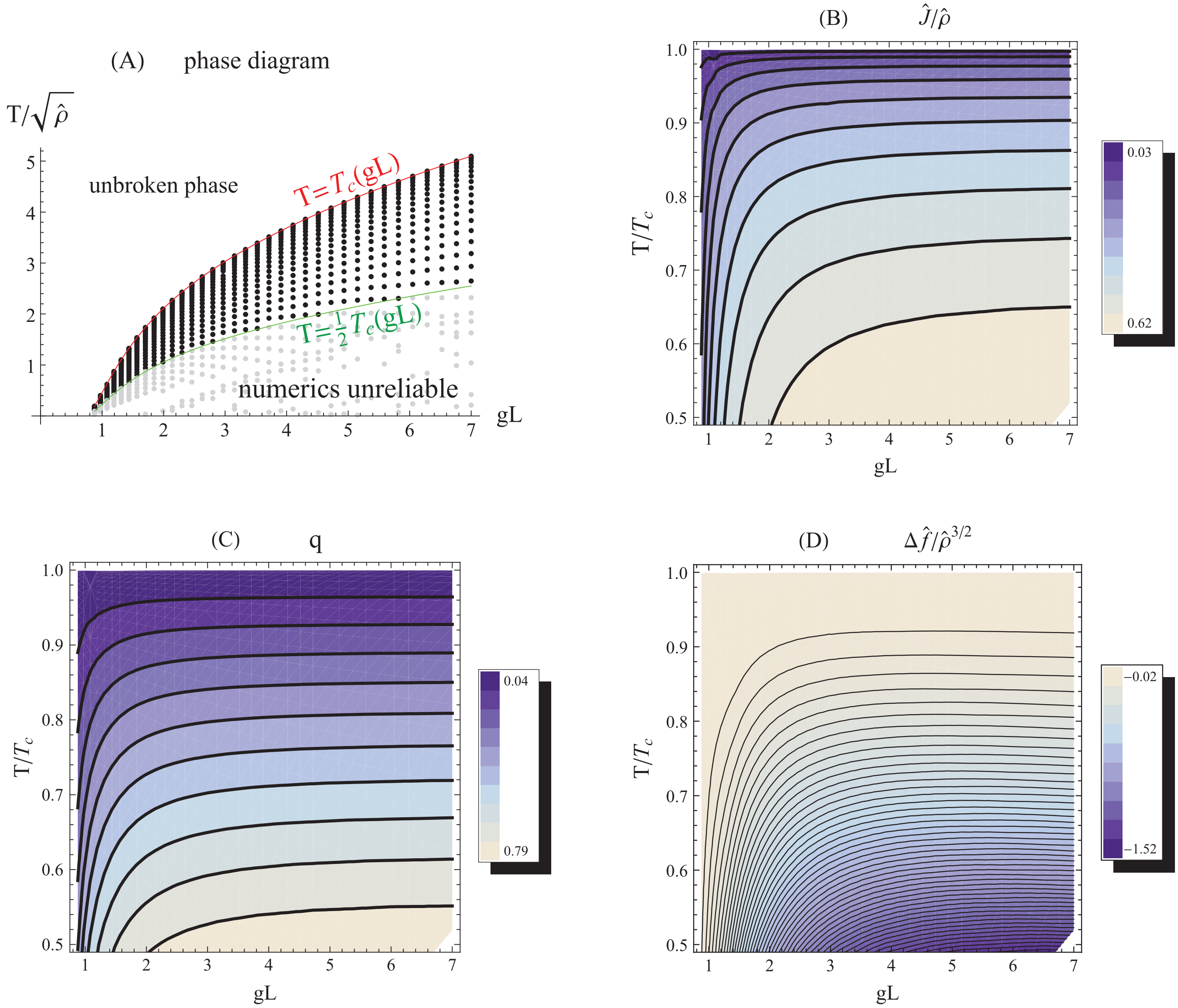

FIG. 1 (color). (a) The phase diagram. Each point corresponds to a numerically constructed black hole solution. (b) $\hat{J}/\hat{\rho}$ as a function of T/T_c. (c) q as a function of T/T_c. (d) $\Delta\hat{f}/\hat{\rho}^{3/2}$ as a function of T/T_c.

(c) The superconducting black holes have lower free energy than RNAdS black holes with the same temperature and charge density.

(d) The transition at T_c is second order, with mean field theory exponents: $q \propto t$, $\hat{J}/\hat{\rho} \propto \sqrt{t}$, and $\Delta\hat{f}/\hat{\rho}^{3/2} \propto -t^2$, where $t = 1 - T/T_c$.

(e) For $gL \gtrsim 3.5$, dimensionless quantities such as $\hat{J}/\hat{\rho}$, q, and $\Delta\hat{f}/\hat{\rho}^{3/2}$ exhibit nearly universal behavior as a function of T/T_c from $T = T_c$ down at least to $\frac{1}{2}T_c$. This universality is related to a large g limit where the back reaction of the gauge field on the metric can be neglected.

The evidence for these claims comes from numerical solutions to the differential equations (8) for various choices of parameters. More specifically: to construct a solution, I started by specifying numerical values for g, L, a_0, h_1, Φ_1, and w_0. Higher order terms in the series expansions (10) can be deduced in terms of these quantities. I initialized a numerical integration tool (MATHEMATICA's NDSolve) near $r = 0$ using the series expansions (10). A shooting strategy was then employed (usually based on varying L with gL held fixed) to implement the constraint of normalizability on w. The shooting algorithm was designed to work well near T_c. In practice, it worked well for $T > T_c/2$.

Fits to the expansions (11) allowed me to extract the thermodynamic quantities discussed above. I restricted attention to solutions with w everywhere positive as well as normalizable. Solutions exist in which w has nodes, but they are probably always thermodynamically disfavored because spatial oscillations in w increase energy density. Figure 1 summarizes the thermodynamic properties of the approximately 1600 superconducting black hole solutions that I constructed.

Among solutions preserving the diagonal $U(1)$ generated by simultaneous gauge and spatial rotations, I believe that the ones I have constructed are thermodynamically preferred. However, investigations subsequent to the work reported here [22,23] suggest that solutions with less residual symmetry are preferred over the ones described herein, at least for large gauge coupling.

Because the two-derivative Lagrangian (2) is mostly determined by symmetry principles, the black holes constructed in this Letter provide a particularly clean example of a black hole phase transition. The transition evokes aspects of superconductivity, sharing, in particular, the crucial feature of a spontaneously broken Abelian gauge symmetry. The condensate (16) carries angular momentum, hinting that some analogy with non-BCS-like superconductors might be possible. Such analogies so far have mostly focused on the pseudogap state [7–10], where phase fluctuations destroy superconductivity. Although the underlying degrees of freedom of duals to AdS_4 vacua are typically large N gauge theories—seemingly distant from semirealistic models of superconductors—one might hope that dynamics related to the global symmetries of the Hubbard model and its relatives can be captured, to some extent, by non-Abelian gauge fields in AdS_4.

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