Novel Schemes for Directly Measuring Entanglement of General States

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(Received 23 February 2008; revised manuscript received 12 May 2008; published 7 November 2008)

An intrinsic relation between maximally entangled states and entanglement measures is revealed, which plays a role in establishing connections for different entanglement quantifiers. We exploit the basic idea and propose a framework to construct schemes for directly measuring entanglement of general states. In particular, we demonstrate that rank-1 local factorizable projective measurements, which are achievable with only one copy of an entangled state involved at a time in a sequential way, are sufficient to directly determine the concurrence of an arbitrary two-qubit entangled state.

DOI: 10.1103/PhysRevLett.101.190503 PACS numbers: 03.67.Mn, 03.65.Ud, 03.65.Wj, 03.67.Lx

Introduction.—Quantum entanglement is one of the most significant features of quantum mechanics [1], which has attracted a lot of interest in the burgeoning field of quantum information science and its intersection with many-body physics [2,3]. Entanglement measures play a central role in the theory of entanglement. It is well known that antilinearity from symmetries with time reversal operations is intrinsically nonlocal, which leads to a natural routine to describe and estimate entanglement [4–10]. These entanglement measures based on nonphysical allowed transformations are usually nonlinear functions of the density matrix elements, and thus are difficult to determine directly in experiments. It is worth pointing out that there exists an alternative experimental favorable class of entanglement quantifiers, which are directly defined through the measurable observables [11,12]. The interesting problems are as follows: How and why can these quantities from (anti)symmetric projections serve as entanglement quantifiers? Is there any connection between the above two different classes of entanglement measures?

As far as determining entanglement is concerned, there are two desirable features. The first is about the parametric efficiency issue. It is inefficient and not necessary to obtain all the state parameters as quantum state tomography [13], in particular, when one considers high dimensional and multipartite quantum systems. This concerns not only experimental determining entanglement itself, but is related to the general theoretical problem about extracting information efficiently from an unknown quantum state with the least measurement cost [14]. Second, in many realistic scenarios, entangled particles are shared by two distant parties, Alice and Bob, e.g., long distance quantum communication. It will be valuable that Alice and Bob can measure entanglement with only local operations on individual subsystems and classical communications (LOCC).

The basic ideas of measuring these entanglement measures based on nonphysical allowed transformations directly without state reconstruction [14–20] mainly rely on multiple copies of entangled state; i.e., a number of entangled states need to be present at the same time. This could be difficult for certain physical systems. The requisite experimental components include structure physical approximation (SPA) and interferometer circuit, the implementation of which with only LOCC is a great challenge. One may wonder whether projective observables can also help to determine nonphysical allowed transformation based entanglement measures.

In this Letter, we address the above problems by revealing an intrinsic connection between maximally entangled states (MES) and the definitions of nonphysical allowed transformation based entanglement measures. The connection enables us to find that (anti)symmetric projections can indeed extract the properties of density matrices with nonphysical allowed transformations. This result opens the possibility of establishing relations between various kinds of entanglement quantifiers [6–12]. The connection also allows us to propose a framework based on local projections to design schemes for directly measuring the entanglement quantifiers from nonphysical allowed transformations. We explicitly demonstrate the benefit in determining entanglement of general two-qubit states. The most remarkable feature is that only one copy of entangled state need be present at a time, which is distinct from other schemes using multiple copies of entangled state. As applications of our idea, we elucidate the physics underlying the first noiseless quantum circuit for the Peres-Horodecki separability criterion [4,5,21], which was obtained in [21] through the mathematical analysis of polynomial invariants [18]. Moreover, one can easily construct a circuit to directly measure the realignment properties of quantum states [10].

Connections between MES and entanglement measures.—One useful tool in the entanglement theory is positive but not completely positive map, with antilinear conjugation as the most representative operation. Following the Peres-Horodecki separability criterion [4,5], a lot of entanglement detection methods and measures based on the conjugation of density operator have been established [6–9]. We propose to mathematically implement nonphysical allowed transformations, in particular, antilinear conjugation, with the notation of MES. From an operational viewpoint, MES with appropriate local unitary operations can be associated to (anti)symmetric projective measurements. Thus, our result makes a connection between antilinear conjugation [6–9] and (anti)symmetric projection based entanglement quantifiers [11,12].

Lemma 1 *Given an n-partite operator A on the Hilbert space* $\mathcal{H} = \mathcal{H}_1 \bigotimes \cdots \bigotimes \mathcal{H}_n$, *with the dimension* $\dim(\mathcal{H}_i) = d_i$, *the maximally entangled state of* $d_i \otimes d_i$ *bipartite system is denoted as* $|S_i\rangle = \sum_{s=0}^{d_i-1} |ss\rangle/\sqrt{d_i}$, then

$$(A \otimes \mathbf{I}_{\bar{1}\cdots\bar{n}})|\mathcal{S}\rangle = (\mathbf{I}_{1\cdots n} \otimes A^T)|\mathcal{S}\rangle \quad \text{with } |\mathcal{S}\rangle = \bigotimes_{i=1}^{n} |S_i\rangle_{i\bar{i}}. \tag{1}$$

Moreover, we have

$$\operatorname{tr} A = \left(\prod_{i=1}^{n} d_i\right)\langle\mathcal{S}|(\mathbf{I}_{1\cdots n} \otimes A)|\mathcal{S}\rangle. \tag{2}$$

Equation (1) is the generalization of the fact that a qubit operator can travel through singlets, which has been used to investigate the localizable entanglement properties of valence bond states [22]. Here, from a different perspective, we view A itself as a density matrix ρ rather than an operator on quantum states; Lemma 1 thus indicates that with the notation of MES, we can mathematically implement the (partial) transpose (conjugation) of arbitrary quantum states. Equation (2) is another key point, which enables us to extract the properties of transformed density operators through the projective measurements associated with MES.

Remark 1.—Our idea is quite different from the SPA, in which the transpose of quantum state is approximated by a completely positive map [14,15]. It is worth pointing out that, in Lemma 1, A can be a density operator of arbitrary dimensional multipartite quantum systems.

Theorem 1 *For a general quantum state* ρ *on the Hilbert space* $\mathcal{H} = \mathcal{H}_1 \bigotimes \cdots \bigotimes \mathcal{H}_n$, *we denote the antilinear transformation of* ρ *as* $\tilde{\rho}_u = (U_1 \otimes \cdots \otimes U_n) \times \rho^*(U_1^\dagger \otimes \cdots \otimes U_n^\dagger)$ *and* $|S_{u_i}\rangle = (\mathbf{I} \otimes U_i)|S_i\rangle$. *It can be seen that* [23]

$$(\rho \otimes \rho)|\mathcal{S}_u\rangle = (\mathbf{I}_{1\cdots n} \otimes \rho\tilde{\rho}_u)|\mathcal{S}_u\rangle \quad \text{with } |\mathcal{S}_u\rangle = \bigotimes_{i=1}^{n} |S_{u_i}\rangle_{i\bar{i}}. \tag{3}$$

This will lead to

$$\operatorname{tr}(\rho\tilde{\rho}_u) = \left(\prod_{i=1}^{n} d_i\right)\operatorname{tr}\left[\bigotimes_{i=1}^{n} \mathcal{P}_u^{(i\bar{i})}(\rho \otimes \rho)\right], \tag{4}$$

where U_i *are local unitary operations, and* $\mathcal{P}_u^{(i\bar{i})} = |S_{u_i}\rangle_{i\bar{i}}\langle S_{u_i}|$ *are projections on two copies of the ith subsystem.*

Remark 2.—Theorem 1 can help us to establish connections between antilinearity and (anti)symmetric projections. The result is quite general, e.g., U_i can be arbitrary local unitary operators, and it is applicable for high dimensional situations by using appropriate U_i or reducing the projections of high dimensional bipartite systems into a sum of two-qubit projections [11]. It also provides an intuitive meaning of Wootters' concurrence, which can be linked with the success probability of establishing MES via entanglement swapping following the above theorem.

Novel schemes for measuring entanglement.—Besides the theoretical interest, with the above connection we find that projective observables can help to determine these nonphysical transformation based entanglement quantifiers with much less experimental effort. We first demonstrate how to directly measure the concurrence of general states [6,8,9], and then explicitly illustrate the physics underlying the first noiseless circuit for the Peres-Horodecki separability criterion [4,5,7,21] following the present idea. Finally, we construct a simple circuit for the realignment separability criterion [10].

I. Scheme for directly measuring the concurrence of general states.—The concurrence family of entanglement measures are defined through the eigenvalues λ_j of $\rho\tilde{\rho}_u$ as in Theorem 1. In order to determine these eigenvalues, quantum state tomography needs to obtain $(d_1 \cdots d_n)^2 - 1$ parameters, while direct strategy without state reconstruction only needs to measure the moments $m_k = \sum_j \lambda_j^k$, the number of which is $d_1 \cdots d_n$, and thus it is quadratically efficient.

Lemma 2 *The moments of* $\rho\tilde{\rho}_u$ *can be obtained as follows:*

$$m_k = (d_a d_b)^k \operatorname{tr}\left[(\mathcal{P}^{(a)} \otimes \mathcal{P}^{(b)})V_{a_2\cdots a_{2k}}V_{b_2\cdots b_{2k}}\bigotimes_{i=1}^{2k}(\rho)_{a_ib_i}\right]. \tag{5}$$

$\mathcal{P}^{(s)} = \mathcal{P}_u^{(s_1s_2)} \otimes \cdots \otimes \mathcal{P}_u^{(s_{2k-1}s_{2k})}$, *and* $V_{s_2\cdots s_{2k}}$ *are k-circle permutations* ($s = a, b$).

Proof. The k-cycle permutation $V^{(k)}|\phi_1\rangle|\phi_2\rangle\cdots|\phi_k\rangle = |\phi_k\rangle|\phi_1\rangle\cdots|\phi_{k-1}\rangle$ is the key element for spectrum measurement based on the property $\operatorname{tr}(V^{(k)}\bigotimes_{i=1}^{k} A_i) = \operatorname{tr}(A_k\cdots A_1)$ [24]. As in Lemma 1, using $2k$ copies of entangled state and with the notation of MES, we can mathematically have k copies of $\rho\tilde{\rho}_u$. Thus, one can get Eq. (5) with the above two facts [23]. ■

Remark 3.—For simplicity, we only give the formulations for bipartite systems; Lemma 2, however, is valid for general multipartite states. Our results provide a simple and general framework to design schemes for directly measuring the concurrence family of entanglement measures.

If we use the similar interferometer circuit for spectrum measurement as usual, m_k can be obtained by controlled k-circle permutation and experimental feasible antisymmetric projections, which means that half of controlledswap operations are saved compared with controlled $2k$-circle permutation. Since m_k are real, we do not have to measure the whole interference pattern in order to obtain the visibility [14,15]. Nevertheless, the implementation of interferometer circuit by LOCC is still complicated. The experimental efforts can be reduced if no interferometer circuit is required. We demonstrate the benefit of our framework in the case of general two-qubit states by showing that only rank-1 local factorizable projective measurements are required, which then leads to another interesting feature that we do not have to manipulate a number of entangled states at a time, even the starting point of our scheme is also based on multiple copies of entangled state.

Consider a general two-qubit state ρ, whose entanglement can be quantified by Wootters' concurrence as $C = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$, where λ_i's are the square roots of the eigenvalues of $\rho\tilde{\rho}$ in the decreasing order [6], with $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$. Before proceeding, we first introduce some notations as $|\varphi_0\rangle = \otimes_{i=1}^{k}|S_y\rangle_{2i-1,2i}$, $|\varphi_1\rangle = V_{2,\ldots,2k}|\varphi_0\rangle$ and $|\varphi_2\rangle = -V_{2k-1,2k}|\varphi_1\rangle$, $|\varphi_3\rangle = |\varphi_1\rangle - |\varphi_2\rangle$, where $V_{1,\ldots,l}|\upsilon_1\rangle\cdots|\upsilon_l\rangle = |\upsilon_l\rangle|\upsilon_1\rangle\cdots|\upsilon_{l-1}\rangle$.

Theorem 2 *Given $2k$ copies of general two-qubit state $\varrho_{2k} = \bigotimes_{i=1}^{2k}(\rho)_{a_ib_i}$, the kth moment m_k can be determined by the rank-1 local projective measurements as*

$$m_1 = 4\langle \mathcal{P}_0^{(a)} \otimes \mathcal{P}_0^{(b)}\rangle$$
$$m_k = \tfrac{1}{4}m_1 m_{k-1} + 2^{2k}(\langle \mathcal{P}_1^{(a)} \otimes \mathcal{P}_1^{(b)}\rangle - \langle \mathcal{P}_2^{(a)} \otimes \mathcal{P}_2^{(b)}\rangle), \tag{6}$$

where $k = 2, 3, 4$, and $\mathcal{P}_0 = |\varphi_0\rangle\langle\varphi_0|$, $\mathcal{P}_l = |\phi_l\rangle\langle\phi_l|$ $(l = 1, 2)$ with $|\phi_1\rangle = (|\varphi_0\rangle + |\varphi_3\rangle)/2$, $|\phi_2\rangle = (|\varphi_0\rangle + i|\varphi_3\rangle)/2$. In particular, for $k = 2$ the expectation value $\langle \mathcal{P}_1^{(a)} \otimes \mathcal{P}_1^{(b)}\rangle = m_1^2/16$.

Proof We denote $\mathcal{P}_{i,j,u,v} = \langle\varphi_j|\langle\varphi_i|\varrho_{2k}|\varphi_u\rangle|\varphi_v\rangle$. It can be seen that $V_{2k-1,2k}|\varphi_0\rangle = -|\varphi_0\rangle$ and $V_{2k-1,2k}|\varphi_3\rangle = |\varphi_3\rangle$, which leads to $\mathcal{P}_{3,0,3,3} = -\mathcal{P}_{3,0,3,3} = 0$. In a similar way, one gets $\mathcal{P}_{i,j,u,v} = 0$ if the four indices are either 0 or 3 and the number of 0 is odd. After simple calculations, we have $\mathcal{P}_{3,3,0,0} = 4(\langle \mathcal{P}_1^{(a)} \otimes \mathcal{P}_1^{(b)}\rangle - \langle \mathcal{P}_2^{(a)} \otimes \mathcal{P}_2^{(b)}\rangle)$ [23]. Furthermore, we denote $|\psi_0\rangle \equiv |\varphi_1\rangle + |\varphi_2\rangle$, thus $\mathcal{P} = \langle\psi_0|\langle\psi_0|\varrho_{2k}|\varphi_0\rangle|\varphi_0\rangle = m_1 m_{k-1}/2^{2k}$. Therefore, the kth moment is

$$m_k = 2^{2k}\mathcal{P}_{1,1,0,0} = 2^{2k}\tfrac{1}{4}(\mathcal{P} + \mathcal{P}_{3,3,0,0}). \tag{7}$$

We conclude that m_k are measurable by only rank-1 local projective observables as Eq. (6). For the case of $k = 2$, $|\phi_1\rangle = |\varphi_1\rangle$ which means that $\langle \mathcal{P}_1^{(a)} \otimes \mathcal{P}_1^{(b)}\rangle = m_1^2/16$. ■

Remark 4.—Our scheme inherits the quadratic efficiency by directly measuring four moments to determine the concurrence of general two-qubit states. Only rank-1 local projective measurements are required, which is expected to provide more flexibility in the experiments.

II. Noiseless quantum circuit for the Peres separability criteria.—The connection between MES and entanglement measures plays its role not only in the concurrence family, but also in the other scenarios. As an example, the first noiseless network to measure the spectrum of a partial transposed density operator [21] from the structure of polynomial invariants can be recovered from a different perspective. Based on Lemma 1, we can mathematically implement the partial transpose as

$$(\mathbf{I}_1 \otimes \rho_{\bar{1}2} \otimes \mathbf{I}_{\bar{2}})|S\rangle_{1\bar{1}}|S\rangle_{2\bar{2}} = [\mathbf{I}_1 \otimes \mathbf{I}_2 \otimes (\rho^{T_2})_{\bar{1}\,\bar{2}}]|S\rangle_{1\bar{1}}|S\rangle_{2\bar{2}}.$$

We note that $V_{\bar{1}\,\bar{3}}|S\rangle_{1\bar{1}}|S\rangle_{3\bar{3}} = V_{13}|S\rangle_{1\bar{1}}|S\rangle_{3\bar{3}}$; thus with k copies of entangled states we obtain the kth moment of ρ^{T_2} as

$$\mathrm{tr}\,(\rho^{T_2})^k = \mathrm{tr}\Big[V_{\bar{1}\cdots\overline{2k-1}}V^{\dagger}_{2\cdots 2k}\Big(\bigotimes_{l=1}^{k}\rho_{\overline{2l-1}2l}\Big)\Big]. \tag{8}$$

The right-hand side of Eq. (8) are exactly the circuits in [21].

III. Realignment criterion for entanglement detection.—Realignment of density operators, defined as $\mathcal{R}(\rho)_{ij,kl} = \rho_{ik,jl}$, is another important operation to establish separability criteria. Based on the trace norm of $\mathcal{R}(\rho)$, Chen *et. al.* derived a low bound for the concurrence of arbitrary dimensional $d_a \otimes d_b$ bipartite systems [10]. One can mathematically implement the realignment as

$$\begin{aligned} V_1\rho_{12} \otimes \mathbf{I}_{\bar{1}\,\bar{2}}|S\rangle_{1\bar{1}}|S\rangle_{2\bar{2}} &= \mathbf{I}_{12} \otimes [\mathcal{R}(\rho)]_{\bar{1}\,\bar{2}}|S\rangle_{1\bar{1}}|S\rangle_{2\bar{2}} \\ V_2\rho_{12} \otimes \mathbf{I}_{\bar{1}\,\bar{2}}|S\rangle_{1\bar{1}}|S\rangle_{2\bar{2}} &= \mathbf{I}_{12} \otimes [\mathcal{R}^{\dagger}(\rho)]_{\bar{1}\,\bar{2}}|S\rangle_{1\bar{1}}|S\rangle_{2\bar{2}}, \end{aligned} \tag{9}$$

where $V_1 = V_{\bar{1}\,\bar{2}}V_{2\bar{2}}V_{12}$, $V_2 = V_{\bar{1}\,\bar{2}}V_{1\bar{1}}V_{12}$, and $|S\rangle = \sum_{s=0}^{d-1}|ss\rangle$ with $d = \max\{d_a, d_b\}$. Thus, one can easily write the kth moment of $\mathcal{R}(\rho)\mathcal{R}^{\dagger}(\rho)$ as [23]

$$\mathrm{tr}\,[\mathcal{R}(\rho)\mathcal{R}^{\dagger}(\rho)]^k = \mathrm{tr}\Big[\bigotimes_{i=1}^{k}(V_{a_{2i-1}a_{2i}}V_{b_{2i-1}b_{2i-2}})\Big(\bigotimes_{i=1}^{2k}\rho_{a_ib_i}\Big)\Big],$$

with $b_0 = b_{2k}$, which enables us to construct a simple noiseless circuit.

Experimental implementation.—The main feature of our scheme is taking advantage of the feasible (anti)symmetric projections to access the properties of these nonphysical allowed transform based entanglement quantifiers, which offers more flexibility in various kinds of physical systems. We demonstrate in the following how experimental efforts can be reduced in directly measuring the concurrence of general two-qubit entangled states.

One can obtain m_1 through antisymmetric projective measurements on two copies as Eq. (6). By noting that $|\phi_2\rangle = U_{34}|\varphi_1\rangle$, where $U_{34} = i + (1 - i)|S_y\rangle\langle S_y|$, only one extra two-qubit gate for each party is required to determine m_2. This can be achieved in certain physical systems, e.g., optical lattice with engineered nearest neighbor interactions [25]. To determine the higher moments m_3 and m_4, more copies of entangled states are required. One potential physical system is the ensembles of multilevel quantum systems, in which 10–20 qubits can be built in single trapped cloud of ground state atoms [26,27]. The single element in our scheme, i.e., rank-1 local projective

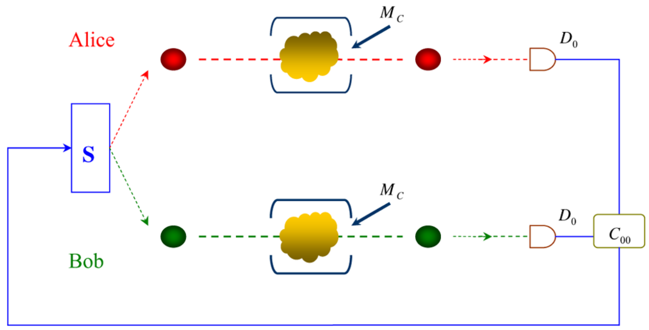

FIG. 1 (color online). Implementation of rank-1 local projective measurements with entangled pairs generated in a sequential way. The preceding measurement results determine whether to generate the remaining copies or to restart the iteration.

measurement, is more favorable in such physical systems than the conventional quantum circuits.

Furthermore, the requirement for the schemes on multiple copies of entangled state [14–20] that a number of entangled state (up to 8 copies) have to be present at the same time becomes unnecessary in our scheme by utilizing the intriguing matrix product state (MPS) formalism [28,29]. Every pure state $|\psi\rangle$ has a MPS representation, and thus can be generated in a sequential way, i.e., $V_{[2k]}\cdots V_{[1]}|\varphi_L\rangle_C|0\cdots 0\rangle_{1\cdots 2k} = |\varphi_R\rangle_C|\psi\rangle_{1\cdots 2k}$, where $|\varphi_L\rangle_C$ and $|\varphi_R\rangle_C$ are the initial and final state of an auxiliary system, e.g., cavity mode or atoms [30]. $V_{[i]}$ is a unitary interaction between qubit i and $\mathcal{C}$. By reversing the above procedure, we can obtain the rank-1 local projective observable $\langle\psi|\langle\psi|\varrho_{2k}|\psi\rangle|\psi\rangle$ of Eq. (6) in a similar sequential way as in Fig. 1. First, the auxiliary system is prepared in $|\varphi_R\rangle$, entangled pairs are generated one by one, pass through and interact with $\mathcal{C}$, then measured along the $\hat{z}$ basis. Only if the results of all steps are 00, we need to measure the auxiliary system with $M_{\mathcal{C}} = |\varphi_L\rangle_C\langle\varphi_L|$. Otherwise, if the result of any step is not 00, we restart the iteration and do not need to generate all the $2k$ copies. Our rough estimation shows that in comparison with quantum state tomography, for each observable, the involved qubits are a little more (5/4 and 4/3 vs 1); however, the total number of entangled states that needs to be generated is even less (95/12 vs 9) [23].

Remark 5—All current schemes for directly measuring entanglement also raise an interesting problem: How does entanglement play its role in reducing the measurement cost in extracting information from an unknown quantum state?

Conclusions.—Maximally entangled state retains its fundamental role in the entanglement theory, which provides an approach to investigate the connections between different entanglement quantifiers. With the notation of maximally entangled states, one can mathematically implement nonphysical allowed transformations of quantum states. This enables us to design novel schemes for directly measuring various kinds of entanglement quantifiers. The benefit is explicitly demonstrated for general two-qubit states, in which only rank-1 local projective measurements are required. It is parametrically efficient without increasing the requirement for state generation over quantum state tomography.

J.-M. C thanks H.-J. Briegel, O. Gühne, and B. Kraus for helpful discussions, and is grateful for support from the FWF through the Lise Meitner Program, and QICS (EU).

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