Irreducible Multiparty Correlations in Quantum States without Maximal Rank

D. L. Zhou

Beijing National Laboratory for Condensed Matter Physics, and Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China

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The correlations of an n-partite quantum state are classified into a series of irreducible k-party ones ($2 \leq k \leq n$), with the irreducible k-party correlation being the correlation in the states of k parties but nonexistent in the states of $(k-1)$ parties. A measure of the degree of irreducible k-party correlation is defined based on the principle of maximal entropy. Adopting a continuity approach, we overcome the difficulties in calculating the degrees of irreducible multiparty correlations for the multipartite states without maximal rank. In particular, we obtain the degrees of irreducible multiparty correlations in the n-qubit stabilizer states and the n-qubit generalized Greenberger-Horne-Zeilinger states, which reveals the distribution of multiparty correlations.

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Introduction.—How to classify and quantify correlations in a multipartite quantum state is a fundamental problem in many-particle physics and quantum information science. Traditionally, this is done by introducing correlation functions between or among experimental observables of different parties. A modern method of characterizing those correlations is based on entropy, in which different types of correlations in a multipartite state are regarded as different types of nonlocal information, and entropy is used as a measure of information [1].

The concept of irreducible n-party correlation in an n-partite quantum state was first proposed in Ref. [2] by Linden *et al.* based on the principle of maximum entropy. This concept describes how much more information in the n-partite state than what is already contained in its reduced states of $(n-1)$ parties. The degree of irreducible two-party correlation in a bipartite quantum state is equal to the two-party mutual entropy [2], which is also obtained in different contexts [3,4]. Among n-qubit pure states, the irreducible n-party correlation is nonzero only for the Greenberger-Horne-Zeilinger (GHZ) type pure states [5]. Most n-partite pure states are completely determined by their reduced states of just over half the parties [6,7].

It is worth pointing out that the idea of using the maximum entropy principle to quantify classical correlation has developed independently in the classical information community [8–10]. Remarkably, the connected information of order k ($2 \leq k \leq n$) for a probability distribution of n classical variables was defined in Ref. [8] by Schneidman *et al.*

In this Letter, we define a measure for the degree of irreducible k-party correlation in an n-partite state. This definition can be regarded as not only a direct generalization of the concept in Ref. [2], but also a quantum version of connected correlation of order k in Ref. [8].

The measure of the degree of irreducible k-party correlation we define relies on a constrained optimization problem over n-partite quantum states. Its explicit calculation for a general n-partite state ($n > 2$) is quite challenging, even for a 3-qubit state. To the best of our knowledge, no explicit calculations for irreducible multiparty correlations exist in the available literature.

The main purpose of this Letter is to calculate the degrees of irreducible multiparty correlations for the multipartite states without maximal rank based on a continuity approach. We obtain analytic results for the degrees of irreducible multiparty correlations of the stabilizer states [11–13] and the generalized GHZ states [5].

Notation and definitions.—Let $[n]$ be the set $\{1, 2, \ldots, n\}$. An m-element subset of $[n]$ is denoted as $\mathbf{a}(m) = \{a_1, a_2, \ldots, a_m\}$, and the relative complement of $\mathbf{a}(m)$ in $[n]$ is denoted as $\bar{\mathbf{a}}(n-m) = \{\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_{(n-m)}\}$.

The state of an n-partite quantum system is specified by an n-partite density matrix $\rho^{[n]}$. The irreducible k-party correlation ($2 \leq k \leq n$) in the state is defined as the information appearing in the k-partite reduced density matrices $\rho^{\mathbf{a}(k)}$, but nonexistent in the $(k-1)$-partite reduced density matrices $\rho^{\mathbf{a}(k-1)}$. To define a measure for the degree of irreducible multiparty correlation in the state $\rho^{[n]}$, we introduce an n-partite density matrix $\tilde{\rho}_l^{[n]}$ for $l = 1, 2, \ldots, n$ as

$$\tilde{\rho}_l^{[n]} = \text{argmax}\{S(\sigma^{[n]}) \colon \sigma^{\mathbf{a}(l)} = \rho^{\mathbf{a}(l)}\} \tag{1}$$

for any subset $\mathbf{a}(l)$, which is similar to the method adopted in Ref. [14]. Function S is the von Neumann entropy defined as $S(\sigma) = -\text{Tr}(\sigma \log_2 \sigma)$. Namely, the n-partite density matrix $\tilde{\rho}_l^{[n]}$ has the same l-partite reduced density matrices as those of $\rho^{[n]}$, but it is maximally noncommittal to other missing information contained in the state $\rho^{[n]}$ [15]. A measure for the degree of irreducible k-party correlation in the state $\rho^{[n]}$ is then defined as

$$C^{(k)}(\rho^{[n]}) = S(\tilde{\rho}_{k-1}^{[n]}) - S(\tilde{\rho}_k^{[n]}). \tag{2}$$

The total correlation in the state $\rho^{[n]}$ is then referred to as the nonlocal information appearing in $\rho^{[n]}$, but nonexistent in the 1-partite states $\rho^{\mathbf{a}(1)}$. A measure of the degree of the total correlation in the state $\rho^{[n]}$ is then defined as

$$C^T(\rho^{[n]}) = S(\tilde{\rho}_1^{[n]}) - S(\tilde{\rho}_n^{[n]}). \tag{3}$$

Substituting Eqs. (2) into Eq. (3), we find that

$$C^T(\rho^{[n]}) = \sum_{k=2}^{n} C^{(k)}(\rho^{[n]}). \tag{4}$$

Equation (4) not only justifies Eq. (3) as a legitimate measure of the total correlation, but also implies that all irreducible multiparty correlations construct a classification of the total correlation. In other words, the degrees of irreducible k-party correlation tell us how the total correlation is distributed in the n-partite quantum state.

As shown in Eqs. (2) and (3), the degrees of different types of correlations are intimately related to the von Neumann entropy. The underlying reason is as follows. The von Neumann entropy of a quantum state is a measure of the degree of uncertainties of the state. The existence of correlation in the multipartite quantum state decreases the uncertainties of the state. Therefore the decrease of uncertainties, i.e., the entropy difference, becomes a reasonable measure for the degree of correlation.

Standard exponential form.—Since the n-partite density matrices $\tilde{\rho}_l^{[n]}$ are essential elements in the definitions in Eqs. (2) and (3), we give the following important Theorem on the standard exponential form of the state $\tilde{\rho}_l^{[n]}$.

Theorem 1.—For an n-partite quantum state $\rho^{[n]}$ with maximal rank, a state $\tilde{\rho}_l^{[n]}$ $(1 \leq l \leq n)$ satisfying Eq. (1) can be expressed in the exponential form

$$\tilde{\rho}_l^{([n])} = \exp\left(\sum_{\mathbf{a}(l)} \Lambda^{\mathbf{a}(l)} \otimes 1^{\bar{\mathbf{a}}(n-l)}\right), \tag{5}$$

where $1^{\bar{\mathbf{a}}(n-l)}$ is the identity operators on the Hilbert space of parties $\bar{\mathbf{a}}(n-l)$, and the operators $\Lambda^{\mathbf{a}(l)}$ are determined by the constrained conditions in Eq. (1).

Proof.—We solve the constrained maximization problem defined by Eq. (1) by the method of Lagrange multipliers.

$$- S(\sigma^{[n]}) - \sum_{\mathbf{a}(l)} \mathrm{Tr}(\Lambda^{\mathbf{a}(l)}(\sigma^{\mathbf{a}(l)} - \rho^{\mathbf{a}(l)})) \geq 1 - \mathrm{Tr}\left(\exp\left(\sum_{\mathbf{a}(l)} \Lambda^{\mathbf{a}(l)} \otimes 1^{\bar{\mathbf{a}}(n-l)}\right)\right) + \mathrm{Tr}(\Lambda^{\mathbf{a}(l)} \rho^{\mathbf{a}(l)}).$$

The equality is satisfied if and only if Eq. (5) is satisfied, and the Lagrange multipliers $\Lambda^{\mathbf{a}(l)}$ are the operators in Eq. (5). To prove the above inequality, we have used the Klein inequality [16]: $\mathrm{Tr}A(\ln A - \ln B) \geq \mathrm{Tr}(A - B)$ for positive operators A and B, where the equality is satisfied if and only if $A = B$. Because the Klein inequality involves only positive operators, we need to limit ourselves to the states with maximal rank. ■

A direct result derived from Theorem 1 is

$$\tilde{\rho}_1^{[n]} = \exp\left(\sum_{\mathbf{a}(1)} \Lambda^{\mathbf{a}(1)} \otimes 1^{\bar{\mathbf{a}}(n-1)}\right) = \prod_{i=1}^{n} \otimes \rho^{(i)}.$$

Therefore the degree of the total correlation (4) in the state $\rho^{[n]}$ is given by

$$C^T(\rho^{[n]}) = \sum_{i=1}^{n} S(\rho^{(i)}) - S(\rho^{[n]}), \tag{6}$$

where we used $\tilde{\rho}_n^{[n]} = \rho^{[n]}$. Although the degree of the total correlation has an analytical expression (6), we failed to find similar analytical results for the degrees of irreducible multiparty correlations $C^{(k)}(\rho^{[n]})$.

Theorem 1 is a direct generalization of Eq. (4) in Ref. [2]. It shows that the feature of multiparty correlation in the state $\tilde{\rho}_l^{[n]}$ is directly embodied in the exponential form of the state. As noted in Ref. [2], Theorem 1 is inapplicable for the multipartite states without maximal rank. However, most multipartite states of interest in many-particle physics or quantum information have nonmaximal ranks, e.g., the n-qubit stabilizer states and the generalized GHZ states to be discussed below.

Our strategy to treat states without maximal rank is based on the fact that a multipartite state without maximal rank can always be regarded as the limit of a series of multipartite states with maximal rank. If the degrees of irreducible multiparty correlations for the series of states with maximal rank can be obtained by using Theorem 1, we can take the limit to get the degrees of irreducible multiparty correlations for the state without maximal rank. We call the above method the continuity approach. The proofs of Theorems 2 and 3 below are typical applications of this approach.

Correlations in stabilizer states.—An n-qubit stabilizer state $\rho_s^{[n]}$ is defined as

$$\rho_s^{[n]} = \frac{1}{2^n} \sum_{\alpha_1,\ldots,\alpha_m=0,1} \prod_{i=1}^{m} g_i^{\alpha_i}, \tag{7}$$

where the operators g_i are m $(m \leq n)$ independent commuting n-qubit Pauli group elements. The set $\mathfrak{g}(\rho_s^{[n]}) = \{g_i\}$ is called the stabilizer generator for the state $\rho_s^{[n]}$, and the group generated by the generator $\mathfrak{g}$, denoted as $\mathcal{G}(\rho_s^{[n]}) = \{\prod_i g_i^{\alpha_i}, \alpha_i = 0, 1\}$, is called the stabilizer of the state. To make Eq. (7) a legitimate state, the minus identity operator is required not to be an element of the stabilizer $\mathcal{G}(\rho_s^{[n]})$. We remark that our definition of the n-qubit stabilizer state is a generalization of the usual definition [13], which corresponds to the case when $m = n$. When $m < n$, the stabilizer states defined by Eq. (7) are no longer pure states.

According to the definition of the n-qubit Pauli group, an element $h \in \mathcal{G}(\rho_s^{[n]})$ can be written as $h = \pm\prod_{i=1}^n O^{(i)}$ for $O \in \{I, X, Y, Z\}$, where I is the 2×2 identity operator, and X, Y, Z are three Pauli matrices. The number of identity operators I in the element h is $N_I(h) = \sum_i \mathrm{Tr}O^{(i)}/2$. The stabilizer $\mathcal{G}(\rho_s^{[n]})$ can be classified into a series of sets $\mathcal{G}_k(\rho_s^{[n]}) = \{h | h \in \mathcal{G}(\rho_s^{[n]}), N_I(h) \geq n-k\}$ for $k \in [n]$. Although in general $\mathcal{G}_k(\rho_s^{[n]})$ is not a group, we can still define a generator $\mathfrak{g}_k(\rho_s^{[n]})$ for the set $\mathcal{G}_k(\rho_s^{[n]})$ as a set of elements in $\mathcal{G}_k(\rho_s^{[n]})$ such that any element in $\mathcal{G}_k(\rho_s^{[n]})$ can be written as a unique product of elements in the set. We remark that $\mathfrak{g}_k(\rho_s^{[n]})$ can be any generator set of the Abelian Pauli subgroup generated by $\mathcal{G}_k(\rho_s^{[n]})$. Even though the generator $\mathfrak{g}_k(\rho_s^{[n]})$ is not uniquely defined, its cardinality $|\mathfrak{g}_k(\rho_s^{[n]})|$ is.

Theorem 2.—The irreducible k-party irreducible correlation in an n-qubit stabilizer state $\rho_s^{[n]}$ is

$$C^{(k)}(\rho_s^{[n]}) = |\mathfrak{g}_k(\rho_s^{[n]})| - |\mathfrak{g}_{k-1}(\rho_s^{[n]})|. \tag{8}$$

Proof.—Because $\mathcal{G}_1(\rho_s^{[n]}) \subseteq \mathcal{G}_2(\rho_s^{[n]}) \subseteq \cdots \subseteq \mathcal{G}_n(\rho_s^{[n]})$, we can always take $\mathfrak{g}_1(\rho_s^{[n]}) \subseteq \mathfrak{g}_2(\rho_s^{[n]}) \subseteq \cdots \subseteq \mathfrak{g}_n(\rho_s^{[n]})$. Then the elements contained in $\mathfrak{g}_k(\rho_s^{[n]})$ but not in $\mathfrak{g}_{k-1}(\rho_s^{[n]})$ are reexpressed as g_{ki} for $i \in [|\mathfrak{g}_k| - |\mathfrak{g}_{k-1}|]$. Thus we can construct an n-qubit state with a real parameter λ as

$$\rho_m^{[n]}(\lambda) = \exp\left(\eta + \lambda \sum_{k=1}^{m} \sum_{i=1}^{|\mathfrak{g}_k|-|\mathfrak{g}_{k-1}|} g_{ki}\right), \tag{9}$$

where $\eta = -\ln(2^n \cosh^{|\mathfrak{g}_m|}\lambda)$, which is determined by the normalization condition $\mathrm{Tr}(\rho_m^{[n]}) = 1$. Then the above state can be expanded as

$$\rho_m^{[n]}(\lambda) = \frac{1}{2^n}\left(1 + \sum_{d=1}^{|\mathfrak{g}_m|} \tanh^d\lambda \sum_{\sum \alpha_{ki}=d} \prod_{k\leq m} g_{ki}^{\alpha_{ki}}\right). \tag{10}$$

Note that if $\exists\ \alpha_{ki} = 1$ for $k > m$, then $\forall\ \mathbf{a}(m)$, we have $\mathrm{Tr}_{\bar{\mathbf{a}}(n-m)}(\prod_{k\leq n} g_{(ki)}^{\alpha_{ki}}) = 0$. Thus the m-partite reduced density matrix $\rho_m^{\mathbf{a}(m)}(\lambda) = \rho_n^{\mathbf{a}(m)}(\lambda)$. According to Theorem 1, the degree of irreducible k-party correlation in the n-qubit state $\rho_n^{[n]}(\lambda)$ is

$$C^{(k)}(\rho_n^{[n]}(\lambda)) = S(\rho_{k-1}^{[n]}(\lambda)) - S(\rho_k^{[n]}(\lambda)). \tag{11}$$

From Eq. (10), we observe that when the parameter λ takes the limit of positive infinity, the states $\rho_m^{[n]}(\lambda)$ are stabilizer states. In particular,

$$\lim_{\lambda\to+\infty} \rho_n^{[n]}(\lambda) = \rho_s^{[n]}. \tag{12}$$

It is easy to prove that $S(\rho_m^{[n]}(+\infty)) = n - |\mathfrak{g}_m(\rho_s^{[n]})|$. For $\lambda \to +\infty$, Eq. (11) becomes Eq. (8). ■

Note that Theorem 2 is consistent with the corresponding result in Ref. [17], which is obtained in terms of a series of (k, n) threshold classical secret sharing protocols.

Theorem 2 can be used to analyze the multiparty correlation distributions in all the stabilizer states. Let us illustrate its power by analyzing the correlations in the two stabilizer states: $\sigma_1^{[3]} = 1/2(|000\rangle\langle 000| + |111\rangle\langle 111|)$ and $\sigma_2^{[3]} = |\mathrm{GHZ}\rangle\langle \mathrm{GHZ}|$ with $|\mathrm{GHZ}\rangle = 1/\sqrt{2}(|000\rangle + |111\rangle)$. For the former, there are 2 bits of correlations altogether, and these 2 bits of correlation are irreducible 2-party correlation. For the latter, the total correlations become 3 bits, and these 3 bits of correlations are classified into 2 bits of irreducible 2-party correlation and 1 bit of irreducible 3-party correlation.

Correlations in generalized GHZ states.—A generalized n-qubit GHZ state is defined as

$$|G_n\rangle = \alpha|00\cdots 0\rangle + \beta|11\cdots 1\rangle, \tag{13}$$

where the parameters α and β satisfy $|\alpha|^2 + |\beta|^2 = 1$ and $\alpha\beta \neq 0$.

Theorem 3.—The degrees of irreducible multiparty correlation in the generalized GHZ state (13) are given by

$$C^{(2)}(\rho_G^{[n]}) = (n-1)H_2(|\alpha|^2), \tag{14}$$

$$C^{(n)}(\rho_G^{[n]}) = H_2(|\alpha|^2), \tag{15}$$

where $\rho_G^{[n]} = |G_n\rangle\langle G_n|$ and $H_2(x) = -x\log_2 x - (1-x)\times \log_2(1-x)$. The degrees of the other types of irreducible multiparty correlation are zero identically, i.e., $C^{(3)}(\rho_G^{[n]}) = C^{(4)}(\rho_G^{[n]}) = \cdots = C^{(n-1)}(\rho_G^{[n]}) = 0$.

Proof.—Let us construct an n-qubit state

$$\rho^{[n]}(\gamma, \vec{\lambda}) = \exp\left(\eta + \gamma \sum_{i=2}^{n} Z^{(1)}Z^{(i)} + \vec{\lambda}\cdot\vec{\Sigma}\right), \tag{16}$$

where the vector $\vec{\lambda} = \lambda_x\hat{x} + \lambda_y\hat{y} + \lambda_z\hat{z} = \lambda\hat{\lambda}$, the operator vector $\vec{\Sigma} = \hat{x}X^{(1)}\prod_{i=2}^n X^{(i)} + \hat{y}Y^{(1)}\prod_{i=2}^n X^{(i)} + \hat{z}Z^{(1)}$, the parameter η is determined by the normalization condition $\mathrm{Tr}(\rho^{[n]}(\gamma, \vec{\lambda})) = 1$, and the notation $\hat{v}$ represents the unit vector along the direction of the vector $\vec{v}$. The $\hat{\lambda}$ component of the operator vector $\vec{\Sigma}$ is denoted as $\Sigma_\lambda = \hat{\lambda}\cdot\vec{\Sigma}$. Note that $\Sigma_\lambda^\dagger = \Sigma_\lambda$, $\Sigma_\lambda^2 = 1$, and $[\Sigma_\lambda, Z^{(1)}Z^{(i)}] = 0$ for $i \in [n]$. The state (16) can thus be written as

$$\rho^{[n]}(\gamma, \vec{\lambda}) = \frac{1}{2^n}\sum_{i=2}^{n}[1 + \tanh(\gamma)Z^{(1)}Z^{(i)}][1 + \tanh(\lambda)\Sigma_\lambda].$$

In the above equation, only the term $Z_1\lambda_z/\lambda$ in Σ_λ contributes to the reduced $(n-1)$-partite reduced density matrices. Therefore the state $\rho^{[n]}(\gamma, \vec{\lambda}')$ has the same $(n-1)$-partite reduced density matrices as the state $\rho^{[n]}(\gamma, \vec{\lambda})$ if the following condition is satisfied:

$$\lambda'_x = \lambda'_y = 0, \qquad \tanh\lambda'_z = \frac{\lambda_z}{\lambda}\tanh\lambda. \tag{17}$$

According to Theorem 1, we find

$$\tilde{\rho}^{[n]}_m(\gamma, \vec{\lambda}) = \rho^{[n]}(\gamma, \vec{\lambda}') \tag{18}$$

for $m = 2, 3, \ldots, n-1$. Therefore the degrees of irreducible multiparty correlations for the state $\rho^{[n]}(\gamma, \vec{\lambda})$ can be obtained via Eqs. (2).

Without loss of generality, we assume that in Eq. (13) $\alpha = \cos(\theta/2)$ and $\beta = \sin(\theta/2)e^{i\phi}$. Then we define the Bloch vector $\hat{u} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Let us take $\hat{\lambda} = \hat{u}$, then $\Sigma_u|G_n\rangle = |G_n\rangle$. The operators $\{Z^{(1)}X^{(i)}\}$ and Σ_u can be regarded as the stabilizer generator of the state $\rho^{[n]}_G$. According to Theorem 2, the relation between the generalized GHZ state $\rho^{[n]}_G$ and $\rho^{[n]}(\gamma, \vec{\lambda})$ is

$$\rho^{[n]}_G = \lim_{\lambda\to+\infty}\rho^{[n]}(\lambda, \lambda\hat{u}). \tag{19}$$

In this case, we find that, for $m = 2, 3, \ldots, n-1$,

$$\lim_{\lambda\to+\infty}\tilde{\rho}^{[n]}_m(\lambda, \lambda\hat{u}) = |\alpha|^2|00\cdots0\rangle\langle00\cdots0| + |\beta|^2|11\cdots1\rangle\langle11\cdots1|. \tag{20}$$

A direct calculation yields the results of Theorem 3. ■

Theorem 3 shows that in the generalized n-qubit GHZ state (13), only irreducible 2-party and n-party correlation exist, and the former is $(n-1)$ times of the latter.

Summary.—The definition of the degree of irreducible k-party correlation in an n-partite state is given as a natural generalization of those defined in [2,8]. The significance of the exponential form of a multipartite state in characterizing irreducible multiparty correlation is emphasized by Theorem 1. Adopting the continuity approach, we are capable of applying Theorem 1 to deal with the irreducible multiparty correlations in multipartite states without maximal rank. Particularly, we successfully obtained the degrees of irreducible k-party correlation in the n-qubit stabilizer states and the n-qubit generalized GHZ states. The multiparty correlation structures in these states are revealed by our results. We hope that the concept of irreducible multiparty correlation will shed light on the characterizations of multiparty correlations in condensed matter system, e.g., topological orders [18–20] in degenerate ground states.

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