Integrable Theory of Quantum Transport in Chaotic Cavities

Vladimir Al. Osipov[1,2] and Eugene Kanzieper[1]

[1]Department of Applied Mathematics, H.I.T.—Holon Institute of Technology, Holon 58102, Israel
[2]Fachbereich Physik, Universität Duisburg-Essen, D-47057 Duisburg, Germany

(Received 17 June 2008; published 22 October 2008)

The problem of quantum transport in chaotic cavities with broken time-reversal symmetry is shown to be completely integrable in the universal limit. This observation is utilized to determine the cumulants and the distribution function of conductance for a cavity with ideal leads supporting an arbitrary number n of propagating modes. Expressed in terms of solutions to the fifth Painlevé transcendent and/or the Toda lattice equation, the conductance distribution is further analyzed in the large-n limit that reveals long exponential tails in the otherwise Gaussian curve.

DOI: 10.1103/PhysRevLett.101.176804 PACS numbers: 73.23.−b, 02.30.Ik, 05.45.Mt

Introduction.—The low temperature electronic conduction through a cavity exhibiting chaotic classical dynamics is governed by quantum phase-coherence effects [1,2]. In the absence of electron-electron interactions [3–5], the most comprehensive theoretical framework by which the phase coherent electron transport can be explored is provided by the scattering $\mathcal{S}$-matrix approach pioneered by Landauer [6]. There exist two different, though mutually overlapping, scattering-matrix descriptions [7] of quantum transport.

A semiclassical formulation [8] of the $\mathcal{S}$-matrix approach is tailor-made to the analysis of energy-averaged charge conduction [9] through an individual cavity. Representing quantum transport observables (such as conductance, shot-noise power, transferred charge, etc.) in terms of classical trajectories connecting the leads attached to a cavity, the semiclassical approach [10] efficiently accounts for system-specific features [11] of the quantum transport. Besides, it also covers the long-time scale universal transport regime [12] emerging in the limit [13] $\tau_D \gg \tau_E$, where τ_D is the average electron dwell time and τ_E is the Ehrenfest time (the time scale where quantum effects set in).

The latter *universal regime* [14] can alternatively be studied within a stochastic approach [4,15] based on a random matrix description [16] of electron dynamics in a cavity. Modeling a single electron Hamiltonian by an $M \times M$ random matrix $\mathcal{H}$ of proper symmetry, the stochastic approach starts with the Hamiltonian $H_{\rm tot}$ of the total system comprised by the cavity and the leads:

$$H_{\rm tot} = \sum_{k,\ell=1}^{M} \boldsymbol{\psi}_k^\dagger \mathcal{H}_{k\ell} \boldsymbol{\psi}_\ell + \sum_{\alpha=1}^{N_L+N_R} \boldsymbol{\chi}_\alpha^\dagger \varepsilon_F \boldsymbol{\chi}_\alpha + \sum_{k=1}^{M} \sum_{\alpha=1}^{N_L+N_R} (\boldsymbol{\psi}_k^\dagger \mathcal{W}_{k\alpha} \boldsymbol{\chi}_\alpha + \boldsymbol{\chi}_\alpha^\dagger \mathcal{W}_{k\alpha}^* \boldsymbol{\psi}_k). \quad (1)$$

Here, $\boldsymbol{\psi}_k$ and $\boldsymbol{\chi}_\alpha$ are the annihilation operators of electrons in the cavity and in the leads, respectively. Indices k and ℓ enumerate electron states in the cavity: $1 \le k, \ell \le M$, with $M \to \infty$. Index α counts propagating modes in the left ($1 \le \alpha \le N_L$) and the right ($N_L + 1 \le \alpha \le N$) lead. The $M \times N$ matrix $\mathcal{W}$ describes the coupling of electron states with the Fermi energy ε_F in the cavity to those in the leads; $N = N_L + N_R$ is the total number of propagating modes (channels). Since in Landauer-type theories the transport observables are expressed in terms of the $N \times N$ scattering matrix [5]

$$\mathcal{S}(\varepsilon_F) = \mathbb{1} - 2i\pi \mathcal{W}^\dagger (\varepsilon_F - \mathcal{H} + i\pi \mathcal{W}\mathcal{W}^\dagger)^{-1} \mathcal{W}, \quad (2)$$

the knowledge of its distribution is central to the stochastic approach. [Two such observables—the conductance $G = {\rm tr}(\mathcal{C}_1 \mathcal{S} \mathcal{C}_2 \mathcal{S}^\dagger)$ and the shot-noise power $P = {\rm tr}(\mathcal{C}_1 \mathcal{S} \mathcal{C}_2 \mathcal{S}^\dagger) - {\rm tr}(\mathcal{C}_1 \mathcal{S} \mathcal{C}_2 \mathcal{S}^\dagger)^2$ measured in proper dimensionless units [4]—are of most interest. Here, $\mathcal{C}_1 = {\rm diag}(\mathbb{1}_{N_L}, 0_{N_R})$ and $\mathcal{C}_2 = {\rm diag}(0_{N_L}, \mathbb{1}_{N_R})$ are the projection matrices].

For random matrices $\mathcal{H}$ drawn from rotationally invariant Gaussian ensembles [17], the distribution of $\mathcal{S}(\varepsilon_F)$ is described [15] by the Poisson kernel [18–20]

$$P(\mathcal{S}) \propto [\det(\mathbb{1} - \bar{\mathcal{S}}\mathcal{S}^\dagger)\det(\mathbb{1} - \mathcal{S}\bar{\mathcal{S}}^\dagger)]^{\beta/2-1-\beta N/2}. \quad (3)$$

Here, β is the Dyson index [17] accommodating system symmetries ($\beta = 1$, 2, and 4) while $\bar{\mathcal{S}}$ is the average scattering matrix [4], $\bar{\mathcal{S}} = V^\dagger {\rm diag}(\sqrt{1-\Gamma_j})V$, that characterizes couplings between the cavity and the leads in terms of tunnel probabilities [21] Γ_j of jth mode in the leads ($1 \le j \le N$); the matrix V is $V \in G(N)/G(N_L) \times G(N_R)$ where G stands for orthogonal ($\beta = 1$), unitary ($\beta = 2$) or symplectic ($\beta = 4$) group.

The above description becomes particularly simple for chaotic cavities that coupled to the leads through ballistic point contacts ("ideal" leads, $\Gamma_j = 1$). Indeed, uniformity of $P(\mathcal{S})$ over $G(N)$ implies that scattering matrices $\mathcal{S}$ belong [22] to one of the three Dyson circular ensembles [17] about which virtually everything is known. Notwithstanding this remarkable simplicity, available analytic results for statistics of electron transport are quite limited [4,23]. In particular, distribution functions of conductance and shot-noise power, as well as their higher order cumulants, are largely unknown for an *arbitrary* number of propagating modes, N_L and N_R, and thus do not catch up with existing experimental capabilities [24].

In this Letter, we combine a stochastic version of the $\mathcal{S}$-matrix approach with ideas of integrability [25,26] to show that the problem of universal quantum transport in chaotic cavities with broken time-reversal symmetry ($\beta = 2$) is completely integrable. Although our theory applies [27] to a variety of transport observables, the further discussion is purposely restricted to the statistics of Landauer conductance. This will help us keep the presentation as transparent as possible.

Conductance distribution.—In order to describe fluctuations of the conductance $G = \mathrm{tr}(\mathcal{C}_1 \mathcal{S} \mathcal{C}_2 \mathcal{S}^\dagger)$ in an adequate way, one needs to know its entire distribution function. To determine the latter, we define the moment generating function

$$\mathcal{F}_n(z) = \langle \exp(-zG) \rangle_{\mathcal{S}\in \mathrm{CUE}(2n+\nu)}, \tag{4}$$

which, in accordance with the above discussion, involves averaging over scattering matrices $\mathcal{S} \in \mathrm{CUE}(2n + \nu)$ drawn from the Dyson circular unitary ensemble [17]. For the sake of convenience, we have introduced the notation $n = \min(N_L, N_R)$ and $\nu = |N_L - N_R|$ so that the total number $N_L + N_R$ of propagating modes in two leads equals $2n + \nu$.

While the averaging in Eq. (4) can explicitly be performed with the help of the Itzykson-Zuber formula [28], a high spectral degeneracy of the projection matrices $\mathcal{C}_1$ and $\mathcal{C}_2$ makes this calculation quite tedious. To avoid unnecessary technical complications, it is beneficial to employ a polar decomposition [19] of the scattering matrix. This brings into play a set of n-transmission eigenvalues $\boldsymbol{T} = (T_1, \cdots, T_n) \in (0, 1)^n$ which characterize the conductance [6] in a particularly simple manner, $G(\boldsymbol{T}) = \sum_{j=1}^n T_j$.

The uniformity of the scattering $\mathcal{S}$-matrix distribution gives rise to a nontrivial joint probability density function of transmission eigenvalues in the form [29,30]

$$P_n(\boldsymbol{T}) = c_n^{-1} \Delta_n^2(\boldsymbol{T}) \prod_{j=1}^n T_j^\nu. \tag{5}$$

Here $\Delta_n(\boldsymbol{T}) = \prod_{j<k}(T_k - T_j)$ is the Vandermonde determinant and c_n is the normalization constant [17]

$$c_n = \prod_{j=0}^{n-1} \frac{\Gamma(j+2)\Gamma(j+\nu+1)\Gamma(j+1)}{\Gamma(j+\nu+n+1)}. \tag{6}$$

Let us stress that the description based on Eq. (5) is completely equivalent to the original, microscopically motivated $\mathcal{S} \in \mathrm{CUE}(2n + \nu)$ model.

Now the moment generating function can elegantly be calculated. A close inspection of the integral

$$\mathcal{F}_n(z) = c_n^{-1} \int_{(0,1)^n} \prod_{j=1}^n dT_j T_j^\nu \exp(-zT_j) \cdot \Delta_n^2(\boldsymbol{T}) \tag{7}$$

reveals that it admits the Hankel determinant representation [25]

$$\mathcal{F}_n(z) = \frac{n!}{c_n} \det[(-\partial_z)^{j+k} \mathcal{F}_1(z)], \tag{8}$$

with

$$\mathcal{F}_1(z) = \frac{(\nu+1)!}{z^{\nu+1}} \left(1 - e^{-z} \sum_{\ell=0}^{\nu} \frac{z^\ell}{\ell!}\right). \tag{9}$$

In deriving Eqs. (8) and (9) we have used the Andréief–de Bruijn integration formula [31].

Equation (8), supplemented by the "initial condition" $\mathcal{F}_0(z) = 1$, has far-reaching consequences. Indeed, by virtue of the Darboux theorem [32], the infinite sequence of the moment generating functions ($\mathcal{F}_1$, $\mathcal{F}_2, \ldots$) obeys the Toda lattice equation ($n \geq 1$)

$$\mathcal{F}_n(z)\mathcal{F}_n''(z) - [\mathcal{F}_n'(z)]^2 = \mathrm{var}_n(G)\mathcal{F}_{n-1}(z)\mathcal{F}_{n+1}(z), \tag{10}$$

where $\mathrm{var}_n(G) = n(n+1)^{-1}(c_{n-1}c_{n+1}/c_n^2)$ is nothing but the conductance variance

$$\mathrm{var}_n(G) = \frac{n^2(n+\nu)^2}{(2n+\nu)^2[(2n+\nu)^2 - 1]}. \tag{11}$$

Since $\mathcal{F}_n(z)$ is the Laplace transform of conductance probability density $f_n(g) = \langle \delta(g - G) \rangle$, the Toda lattice equation provides an exact solution [33] to the problem of conductance distribution in chaotic cavities with an arbitrary number of channels in the leads. Equations (9)–(11) represent the first main result of the Letter.

There exists yet another way to describe the conductance distribution. Spotting that the moment generating function $\mathcal{F}_n(z)$ is essentially a Fredholm determinant [34] associated with a gap formation probability [17] within the interval $(z, +\infty)$ in the spectrum of an auxiliary $n \times n$ Laguerre unitary ensemble,

$$\mathcal{F}_n(z) \propto z^{-n(n+\nu)} \int_{(0,z)^n} \prod_{j=1}^n d\lambda_j \lambda_j^\nu e^{-\lambda_j} \cdot \Delta_n^2(\boldsymbol{\lambda}), \tag{12}$$

one immediately derives [34,35]:

$$\mathcal{F}_n(z) = \exp\left(\int_0^z dt \frac{\sigma_{\mathrm{V}}(t) - n(n+\nu)}{t}\right). \tag{13}$$

Here, $\sigma_{\mathrm{V}}(t)$ satisfies the Jimbo-Miwa-Okamoto form of the Painlevé V equation [36]

$$(t\sigma_{\mathrm{V}}'')^2 + [\sigma_{\mathrm{V}} - t\sigma_{\mathrm{V}}' + 2(\sigma_{\mathrm{V}}')^2 + (2n+\nu)\sigma_{\mathrm{V}}']^2 + 4(\sigma_{\mathrm{V}}')^2(\sigma_{\mathrm{V}}' + n)(\sigma_{\mathrm{V}}' + n + \nu) = 0 \tag{14}$$

subject to the boundary condition $\sigma_{\mathrm{V}}(t \to 0) \simeq n(n+\nu)$.

To the best of our knowledge, this is the first ever appearance of Painlevé transcendents in problems of quantum transport. The representation Eq. (13), being the second main result of the Letter, opens a way for a nonperturbative calculation of conductance cumulants.

Conductance cumulants.—Our third main result is the bilinear recurrence relation ($j \geq 2$)

$$[(2n+\nu)^2 - j^2](j+1)\kappa_{j+1} = 2\sum_{\ell=0}^{j-1}(3\ell+1)(j-\ell)^2\binom{j+1}{\ell+1}\kappa_{\ell+1}\kappa_{j-\ell} - (2n+\nu)(2j-1)j\kappa_j - j(j-1)(j-2)\kappa_{j-1} \tag{15}$$

for conductance cumulants $\{\kappa_j\}$. Taken together with the initial conditions provided by the average conductance $\kappa_1 = n(n+\nu)/(2n+\nu)$ and the conductance variance $\kappa_2 = \kappa_1^2/[(2n+\nu)^2 - 1]$, this recurrence efficiently generates (previously unavailable) conductance cumulants of *any* given order.

To prove Eq. (15), we compare Eq. (13) with the definition of the cumulant generating function

$$\log \mathcal{F}_n(z) = \sum_{j=1}^{\infty}\frac{(-1)^j}{j!}\kappa_j z^j \tag{16}$$

to deduce the remarkable identity

$$\sigma_{\rm V}(z) = n(n+\nu) + \sum_{j=1}^{\infty}\frac{(-1)^j}{(j-1)!}\kappa_j z^j. \tag{17}$$

Substituting it back to Eq. (14), we discover Eq. (15) as well as the above stated initial conditions.

Large-n limit of the theory.—The nonpeturbative solution Eq. (15) has a drawback: it does not supply much desired *explicit* dependence of conductance cumulants κ_j's on j. To probe the latter, we turn to the large-n limit of the recurrence Eq. (15). For simplicity, the asymmetry parameter ν will be set to zero.

Since, in the limit of a large number of propagating modes ($n \gg 1$), the conductance distribution is expected [37] to follow the Gaussian law

$$f_n^{(0)}(g) = \frac{1}{\sqrt{2\pi \mathrm{var}_\infty(G)}}\exp\left(-\frac{(g-n/2)^2}{2\mathrm{var}_\infty(G)}\right) \tag{18}$$

with the average conductance $\mathbb{E}[G] = n/2$ and the conductance variance $\mathrm{var}_\infty(G) = 1/16$, it is natural to seek a large-n solution to Eq. (15) in the form $\kappa_j = (n/2)\delta_{j,1} + (1/16)\delta_{j,2} + \delta\kappa_j$, where $\delta\kappa_j$ (with $j \geq 3$) account for deviations from the Gaussian distribution. Next, we put forward the large-n ansatz

$$\delta\kappa_j = \frac{1}{n^j}\sum_{m=0}^{\infty}\frac{a_m(j)}{n^m}, \tag{19}$$

which, after its substitution into the recurrence, yields the explicit formula

$$\delta\kappa_{2j} = \frac{1}{4}\frac{(2j-1)!}{(4n)^{2j}}\left[1 + \frac{j(3j^2-1)}{8n^2} + \mathcal{O}\left(\frac{1}{n^4}\right)\right]. \tag{20}$$

All odd order cumulants vanish identically.

Interestingly, Eq. (20) makes it possible to analytically study a deviation of conductance distribution $f_n(g)$ from the Gaussian law $f_n^{(0)}(g)$. The Gram-Charlier expansion

$$f_n(g) = \exp\left(\sum_{j=1}^{\infty}\frac{\delta\kappa_j}{j!}(-\partial_g)^j\right)f_n^{(0)}(g) \tag{21}$$

is the key. As soon as $|\partial_g \log f_n^{(0)}(g)| \sim n$, the operator in the exponent is dominated by the $m = 0$ term in Eq. (19). This observation reduces Eq. (21) to

$$f_n(g) = \frac{2n^{1/4}}{\Gamma(1/8)}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{d\lambda e^{-n^2\lambda}}{\lambda^{7/8}\sqrt{1+2\lambda}}\exp\left(-\frac{2n^2\eta^2}{1+2\lambda}\right). \tag{22}$$

Here, η is the rescaled conductance $\eta = 2(g/n) - 1$.

Equation (22) is particularly suitable for the asymptotic analysis. Performed with a logarithmic accuracy, it brings

$$\log f_n(g) \sim \begin{cases} -2n^2\eta^2, & |\eta| < \frac{1}{2} \\ -2n^2(|\eta| - \frac{1}{4}) - \frac{3}{4}\log n, & \frac{1}{2} < |\eta| < 1 \end{cases}. \tag{23}$$

This result shows that the Gaussian approximation for the conductance distribution is only valid for $|g - n/2| < n/4$. Away from this region, the conductance distribution exhibits long tails described by the exponential rather than the Gaussian law. Finally, it is straightforward to derive from the Toda lattice Eq. (10) that, in the vicinity $|g - g_*| \leq 1$ of the edges [33] $g_* = 0$ and $g_* = n$, the conductance distribution exhibits even slower, power-law decay [20,23]

$$\log f_n(g) \sim (n^2 - 1)\log(2|\eta - \eta_*|) - \frac{n^2}{2} + \frac{1}{12}\log n \tag{24}$$

with $\eta_* = \pm 1$.

Conclusions.—We have shown that a marriage between the scattering $\mathcal{S}$-matrix approach and the theory of integrable systems brings out an efficient formalism tailor made to analysis of the universal aspects of quantum transport in chaotic systems with broken time-reversal symmetry. Having chosen the paradigmatic problem of conductance fluctuations in chaotic cavities with ideal leads as an illustrative example, we determined the cumulants of conductance as well as its distribution exactly for any given number of propagating modes in the leads. It should be stressed that the ideas presented in the Letter can equally be utilized [27] to describe statistical properties of the shot-noise power and the dynamics of charge transfer.

Certainly, more effort is needed to accomplish integrable theory of the universal quantum transport. Extension of the formalism presented to the $\beta = 1$ and 4 symmetry classes and waiving the uniformity of the $\mathcal{S}$-matrix distribution are the two most challenging problems whose solution is very much called for.

This work was supported by the Israel Science Foundation through the Grant No 286/04.

Note added.—Recently, we learned about the paper by M. Novaes [38] who noticed that the nth *moment* of conductance can nonperturbatively be calculated by using the machinery of hypergeometric functions of matrix argument. Neither Toda lattice nor Painlevé V representations for the conductance distribution surfaced there.

[1] *Mesoscopic Quantum Physics*, edited by E. Akkermans, G. Montambaux, J.-L. Pichard, and J. Zinn-Justin (Elsevier, Amsterdam, 1995).
[2] Y. Imry, *Introduction to Mesoscopic Physics* (Oxford University Press, New York, 2002).
[3] A description in terms of noninteracting electrons is justified for chaotic cavities with sufficiently large capacitance; see, e.g., Ref. [4].
[4] C. W. J. Beenakker, Rev. Mod. Phys. **69**, 731 (1997).
[5] Y. Alhassid, Rev. Mod. Phys. **72**, 895 (2000).
[6] R. Landauer, IBM J. Res. Dev. **1**, 223 (1957); D. S. Fisher and P. Lee, Phys. Rev. B **23**, R6851 (1981); M. Büttiker, Phys. Rev. Lett. **65**, 2901 (1990).
[7] C. H. Lewenkopf and H. A. Weidenmüller, Ann. Phys. (N.Y.) **212**, 53 (1991).
[8] K. Richter, *Semiclassical Theory of Mesoscopic Quantum Systems* (Springer, New York, 2000).
[9] Energy averaging is performed over such a small energy window near the Fermi energy that keeps the classical dynamics essentially unchanged.
[10] İ. Adagideli, Phys. Rev. B **68**, 233308 (2003); R. S. Whitney and P. Jacquod, Phys. Rev. Lett. **96**, 206804 (2006); P. W. Brouwer and S. Rahav, Phys. Rev. B **74**, 075322 (2006); P. W. Brouwer, Phys. Rev. B **76**, 165313 (2007).
[11] I. L. Aleiner and A. I. Larkin, Phys. Rev. B **54**, 14423 (1996); Phys. Rev. E **55**, R1243 (1997); O. Agam, I. Aleiner, and A. Larkin, Phys. Rev. Lett. **85**, 3153 (2000).
[12] K. Richter and M. Sieber, Phys. Rev. Lett. **89**, 206801 (2002); S. Heusler, S. Müller, P. Braun, and F. Haake, Phys. Rev. Lett. **96**, 066804 (2006); P. Braun, S. Heusler, S. Müller, and F. Haake, J. Phys. A **39**, L159 (2006); S. Müller, S. Heusler, P. Braun, and F. Haake, New J. Phys. **9**, 12 (2007).
[13] The Ehrenfest time $\tau_E \simeq \lambda^{-1}\log(W/\lambda_F)$ is determined by the Lyapunov exponent λ of chaotic classical dynamics, the Fermi wavelength λ_F, and the lead widths W. The mean dwell time $\tau_D \simeq A/(Wv_F)$, where A is the area of the cavity.
[14] The notion of universality should be taken with some care since transport observables will generically depend on nonuniversal couplings between the cavity and the leads; see Eqs. (1)–(3).
[15] P. W. Brouwer, Phys. Rev. B **51**, 16878 (1995).
[16] O. Bohigas, M. J. Giannoni, and C. Schmit, Phys. Rev. Lett. **52**, 1 (1984).
[17] M. L. Mehta, *Random Matrices* (Elsevier, Amsterdam, 2004).
[18] The celebrated result Eq. (3), that can be viewed as a generalization [19] of the three Dyson circular ensembles [17], was alternatively derived through a phenomenological information-theoretic approach reviewed in Ref. [20].
[19] L. K. Hua, *Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains* (American Mathematical Society, Providence, Rhode Island, 1963).
[20] P. A. Mello and H. U. Baranger, Waves Random Media **9**, 105 (1999).
[21] The tunnel probabilities $\Gamma_j = 4\tilde{w}_j/(1+\tilde{w}_j)^2$ are determined by the eigenvalues $\tilde{w}_j$ of the $M \to \infty$ matrix $(\pi^2/M\Delta)\mathcal{W}^\dagger\mathcal{W}$, where Δ is the mean level spacing. The average electron dwell time $\tau_D = (2\pi\hbar/\Delta) \times (\sum_{j=1}^N \Gamma_j)^{-1}$.
[22] R. Blümel and U. Smilansky, Phys. Rev. Lett. **64**, 241 (1990).
[23] For most general results available, see: H.-J. Sommers, W. Wieczorek, and D. V. Savin, Acta Phys. Pol. A **112**, 691 (2007); D. V. Savin, H.-J. Sommers, and W. Wieczorek, Phys. Rev. B **77**, 125332 (2008); P. Vivo and E. Vivo, J. Phys. A **41**, 122004 (2008).
[24] S. Oberholzer, E. V. Sukhorukov, C. Strunk, C. Schönenberger, T. Heinzel, and M. Holland, Phys. Rev. Lett. **86**, 2114 (2001); S. Oberholzer, E. V. Sukhorukov, and C. Schönenberger, Nature (London) **415**, 765 (2002).
[25] E. Kanzieper, Phys. Rev. Lett. **89**, 250201 (2002); E. Kanzieper, in: *Frontiers in Field Theory*, edited by O. Kovras (Nova Science Publishers, New York, 2005).
[26] V. Al. Osipov and E. Kanzieper, Phys. Rev. Lett. **99**, 050602 (2007).
[27] V. Al. Osipov and E. Kanzieper (unpublished).
[28] C. Itzykson and J. B. Zuber, J. Math. Phys. (N.Y.) **21**, 411 (1980).
[29] H. U. Baranger and P. A. Mello, Phys. Rev. Lett. **73**, 142 (1994).
[30] P. J. Forrester, J. Phys. A **39**, 6861 (2006).
[31] C. Andréief, Mém. Soc. Sci. **2**, 1 (1883); N. G. de Bruijn, J. Indian Math. Soc. **19**, 133 (1955).
[32] G. Darboux, *Lecons sur la Theorie generale des Surfaces et les Applications Geometriques du Calcul Infinitesimal* (Chelsea, New York, 1972), Vol. II, p. XIX.
[33] Because of a specific form of $\mathcal{F}_1(z)$, a calculation of the inverse Laplace transform of $\mathcal{F}_n(z)$ is straightforward. The resulting probability density function $f_n(g)$ is seen to be a nonanalytic function with a finite support $g \in (0, n)$.
[34] C. A. Tracy and H. Widom, Commun. Math. Phys. **163**, 33 (1994).
[35] P. J. Forrester and N. S. Witte, Commun. Pure Appl. Math. **55**, 679 (2002).
[36] M. Jimbo, T. Miwa, Y. Môri, and M. Sato, Physica (Amsterdam) **1D**, 80 (1980); K. Okamoto, Japanese J. Math. **13**, 47 (1987).
[37] H. D. Politzer, Phys. Rev. B **40**, 11 917 (1989).
[38] M. Novaes, Phys. Rev. B **78**, 035337 (2008).