Next-to-Next-to-Leading Order Corrections to Three-Jet Observables in Electron-Positron Annihilation

Stefan Weinzierl
Institut für Physik, Universität Mainz, D-55099 Mainz, Germany
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I report on a numerical program, which can be used to calculate any infrared safe three-jet observable in electron-positron annihilation to next-to-next-to-leading order in the strong coupling constant α_s. The results are compared to a recent calculation by another group. Numerical differences in three color factors are discussed and explained.

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Introduction.—Jet observables and event shapes in electron-positron annihilation can be used to extract the value of the strong coupling constant α_s [1–3]. This applies, in particular, to three-jet observables, where the leading-order parton process is proportional to α_s. In order to extract the numerical value from the LEP data, precise theoretical calculations are necessary, calling for a next-to-next-to-leading order (NNLO) calculation. Because of the large variety of interesting jet observables it is desirable not to perform this calculation for a specific observable, but to set up a computer program, which yields predictions for any infrared safe observable relevant to the process $e^+e^- \to 3$ jets. Such a task requires the calculation of the relevant amplitudes up to two loops, a method for the cancellation of infrared divergences and stable and efficient Monte Carlo techniques. For the process $e^+e^- \to 2$ jets this was done in [4–7]. In this Letter I report on a NNLO calculation for three-jet observables in electron-positron annihilation. Recently, another group published results for the NNLO corrections for three-jet observables [8–11]. In the calculation presented here the methods used are in many parts similar to the ones used in [8–11], although I will show that at certain points there are important differences. The authors of [8–11] made major contributions to the development of these methods [5,12–15].

The numerical results of the two calculations are compared. The comparison is facilitated by splitting the NNLO correction term into individually gauge-invariant contributions, such that each contribution is proportional to a specific color factor. For the NNLO corrections to $e^+e^- \to 3$ jets there are six different color factors. In three color factors the two calculations agree (N_c^{-2}, N_f/N_c, N_f^2). They disagree in the remaining three color factors (N_c^2, N_c^0, $N_f N_c$,). The numerical differences in these color factors can be traced back to an incomplete cancellation of soft-gluon singularities in the calculation of Refs. [8–11]. These singularities require additional subtraction terms, which are subtracted from the five-parton configuration and added to the four-parton configuration. These subtraction terms have a structure not present in [10] and are related to soft gluons. These terms occur generically in any NNLO calculation with three or more hard-colored partons.

General setup.—The perturbative expansion of any infrared-safe observable for the process $e^+e^- \to 3$ jets can be written up to NNLO as

$$\mathcal{O} = \frac{\alpha_s}{2\pi} A_{\mathcal{O}} + \left(\frac{\alpha_s}{2\pi}\right)^2 B_{\mathcal{O}} + \left(\frac{\alpha_s}{2\pi}\right)^3 C_{\mathcal{O}}. \tag{1}$$

$A_{\mathcal{O}}$ gives the LO result, $B_{\mathcal{O}}$ the NLO correction and $C_{\mathcal{O}}$ the NNLO correction. The coefficient $C_{\mathcal{O}}$ can be decomposed into color pieces

$$C_{\mathcal{O}} = \frac{1}{8}(N_c^2 - 1)\left[N_c^2 C_{\mathcal{O}}^{lc} + C_{\mathcal{O}}^{sc} + \frac{1}{N_c^2} C_{\mathcal{O}}^{ssc} + N_f N_c C_{\mathcal{O}}^{nf} + \frac{N_f}{N_c} C_{\mathcal{O}}^{nfsc} + N_f^2 C_{\mathcal{O}}^{nfnf}\right], \tag{2}$$

where N_c denotes the number of colors and N_f the number of light quark flavours. In addition, there are singlet contributions, which arise from interference terms of amplitudes, where the electroweak boson couples to two different fermion lines. These singlet contributions are expected to be numerically small [16–18] and neglected in the present calculation.

The computation of the NNLO coefficient $C_{\mathcal{O}}$ requires the knowledge of the amplitudes for the three-parton final state $e^+e^- \to \bar{q}qg$ up to two loops [18,19], the amplitudes of the four-parton final states $e^+e^- \to \bar{q}qgg$ and $e^+e^- \to \bar{q}q\bar{q}q$ up to one-loop [20–23] and the five-parton final states $e^+e^- \to \bar{q}qggg$ and $e^+e^- \to \bar{q}q\bar{q}qg$ at tree level [24,25]. Taken separately, the three-, four-, and five-parton contributions are all individually infrared divergent. Only the sum of them is finite. However, the individual contributions live on different phase spaces, which prevents a naive Monte Carlo approach. To render the individual contributions finite, several options for the cancellation of infrared divergences have been discussed, like phase space slicing [26], sector decomposition [27,28], a method based on the optical theorem [29] or the subtraction method [5,12–15,30–41]. In the present calculation I use the subtraction method with antenna subtraction terms [15].

Cancellation of divergences.—To render the individual three-, four-, and five-parton contributions finite, one adds and subtracts suitable chosen terms. Schematically, we have

$$\text{5 partons: } d\sigma_5^{(0)} - d\alpha^{\text{NLO}} - d\alpha^{\text{NNLO}} + d\alpha^{\text{iterated}} - d\alpha^{\text{almost}} - d\alpha^{\text{soft}},$$

$$\text{4 partons: } d\sigma_4^{(1)} + d\alpha^{\text{NLO}} - d\alpha^{\text{loop}} - d\alpha^{\text{iterated}} - d\alpha^{\text{product}} + d\alpha^{\text{almost}} + d\alpha^{\text{soft}},$$

$$\text{3 partons: } d\sigma_3^{(2)} + d\alpha^{\text{NNLO}} + d\alpha^{\text{loop}} + d\alpha^{\text{product}}.$$

Here, $d\sigma_5^{(0)}$, $d\sigma_4^{(1)}$, and $d\sigma_3^{(2)}$ are the contributions from the original amplitudes with five, four or three final state partons. $d\alpha^{\text{NLO}}$ is the NLO subtraction term for four-jet observables, containing only three-parton tree-level antenna functions. At NNLO there are several new subtraction terms required, each of them with a specific structure. The term $d\alpha^{\text{NNLO}}$ contains the four-parton tree-level antenna functions. The term $d\alpha^{\text{loop}}$ contains three-parton one-loop antenna functions together with tree-level matrix elements and three-parton tree-level antenna functions together with one-loop matrix elements. The remaining terms $d\alpha^{\text{iterated}}$, $d\alpha^{\text{almost}}$, $d\alpha^{\text{product}}$, and $d\alpha^{\text{soft}}$ all contain a product of two three-parton tree-level antenna functions. In $d\alpha^{\text{iterated}}$ and $d\alpha^{\text{almost}}$ one antenna function has five-parton kinematics, while the other antenna has four-parton kinematics. The former subtraction term is an approximation to $d\alpha^{\text{NLO}}$, while the latter approximates $d\sigma_5^{(0)}$ in almost color-correlated double unresolved configurations. In $d\alpha^{\text{product}}$ both antennas have four-parton kinematics. The term $d\alpha^{\text{soft}}$ will be discussed below and is relevant only for the color factors N_c^2, N_c^0, and $N_f N_c$.

The subtraction terms without $d\alpha^{\text{soft}}$ correspond to the subtraction scheme of Ref. [10]. For any subtraction scheme it is required, that in the three-parton channel the explicit divergences cancel, that the four-parton channel is integrable over a single unresolved phase space and in addition that the explicit divergences cancel and finally that in the five-parton channel the integrand is integrable over single and double unresolved phase space regions. It is easily checked that with the subtraction terms of Ref. [10] the explicit divergences in the three-parton cancel and I will focus in the following on the four- and five-parton channels.

In the four-parton channel the combination $d\sigma_4^{(1)} + d\alpha^{\text{NLO}}$ is free of explicit poles. It has been noted in Ref. [10] that the combination $d\alpha^{\text{loop}} + d\alpha^{\text{iterated}} + d\alpha^{\text{product}} - d\alpha^{\text{almost}}$ involves in the color factors N_c^2 and N_c^0 poles of the form

$$|\mathcal{A}_3^{(0)}(1', 2', j)|^2 X_3^0(1, i, 2) \frac{1}{\varepsilon}\left[\ln\frac{s_{1'j}s_{j2'}}{s_{1'2'}} - \ln\frac{s_{1j}s_{j2}}{s_{12}}\right],$$

where $p_{1'}$ and $p_{2'}$ are the momenta obtained from p_1, p_i and p_2 through a $3 \to 2$ phase space map. $\mathcal{A}_3^{(0)}$ is the three-parton tree-level amplitude and $X_3^0(1, i, 2)$ a three-parton tree-level antenna function. In Ref. [10] it was claimed that these poles vanish after the azimuthal integration over the unresolved phase space. This claim is wrong. In the center-of-mass frame of $p_{1'} + p_{2'}$ with $p_{1'}$ and p_1 along the positive z axis, the relevant integral is

$$I = \frac{1}{2\pi}\int_0^{2\pi} d\phi \ln\left(\frac{(1 + c_j)(1 - c_2)}{2(1 - c_2 c_j - s_2 s_j \cos\phi)}\right), \quad (3)$$

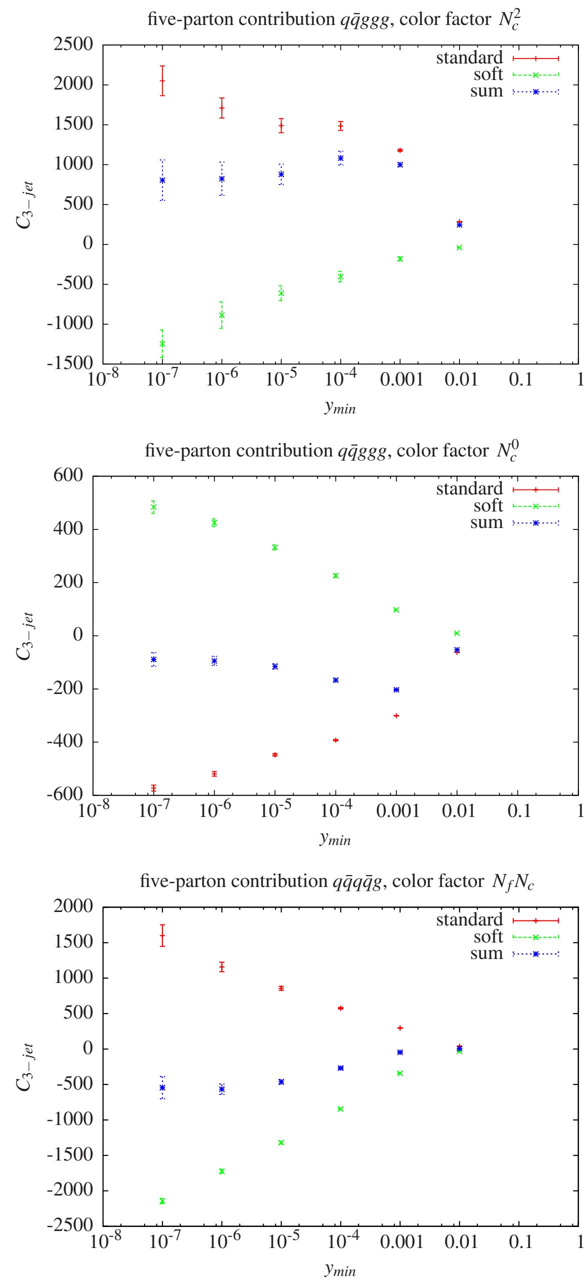

FIG. 1 (color online). Dependence of the five-parton contribution on the slicing parameter y_{min} for the Durham jet cross section with $y_{\text{cut}} = 0.01$ in the color factors N_c^2, N_c^0 and $N_f N_c$. "Standard" denotes the combination $d\sigma_5^{(0)} - d\alpha^{\text{NLO}} - d\alpha^{\text{NNLO}} + d\alpha^{\text{iterated}} - d\alpha^{\text{almost}}$, "soft" the contribution from $d\alpha^{\text{soft}}$. In addition the sum of the two terms is shown. For small values of y_{min} the sum is independent of y_{min}.

where for $x = 2, j$ we set $c_x = \cos\theta_x$, $s_x = \sin\theta_x$ and θ_2 and θ_j are the polar angles of partons 2 and j in the chosen frame. The integral equals

$$I = \ln\left(\frac{1 - c_2 c_j + (c_j - c_2)}{1 - c_2 c_j + |c_j - c_2|}\right). \quad (4)$$

The integral is zero for $\theta_j < \theta_2$ but nonzero for $\theta_j > \theta_2$. In Ref. [10] it was claimed that the integral vanishes in both cases. As a consequence of the nonzero value for $\theta_j > \theta_2$ the explicit poles do not cancel in the combination $d\alpha^{\text{loop}} + d\alpha^{\text{iterated}} + d\alpha^{\text{product}} - d\alpha^{\text{almost}}$. The same situation occurs also in the color factor $N_f N_c$.

These poles have a counterpart in the five-parton channel. Setting $d\alpha^{\text{soft}}$ to zero and using a slicing approach, one observes in the color factors N_c^2, N_c^0, and $N_f N_c$ a logarithmic dependence on the slicing parameter $y_{\min} = s_{\min}/Q^2$. This is shown in Fig. 1.

These singularities require an additional subtraction term and that is where the present calculation differs from the one of Refs. [8–11]. $d\alpha^{\text{soft}}$ is a subtraction term related to soft gluons which ensures that the poles in the four-parton configuration vanish after integration over the azimuthal angle and which renders the five-parton configuration independent of $y_{\min}$. The term $d\alpha^{\text{soft}}$ for the four-parton configuration can be taken of the form

$$|\mathcal{A}_3^{(0)}(1', 2', j)|^2 X_3^0(1, i, 2)\theta\left(\frac{2p_1 p_j}{2p_{1i2} p_j} - \frac{2p_1 p_2}{2p_{1i2} p_2}\right) \times [\mathcal{S}_3^0(s_{1j}) - \mathcal{S}_3^0(s_{12}) - \mathcal{S}_3^0(s_{\hat{2}j}) + \mathcal{S}_3^0(s_{\hat{2}2})], \quad (5)$$

where $\mathcal{S}_3^0$ is the integrated soft antenna function and $p_{\hat{2}}$ is given by

$$p_{\hat{2}} = p_2 + p_i - \frac{s_{2i}}{s_{12} + s_{1i}} p_1. \quad (6)$$

I also used the shorthand notation $p_{1i2} = p_1 + p_i + p_2$. The θ-function enforces $\theta_j > \theta_2$ in the specific frame introduced above. $d\alpha^{\text{soft}}$ for the five-parton configuration is obtained by lifting Eq. (5) to the five-parton phase space. Figure 1 shows that the sum of all contributions in the five-parton channel is now independent of $y_{\min}$. I have checked that in the four-parton channel the explicit poles cancel after integration over the unresolved phase space.

Numerical results.—The numerical program is built on an existing NLO program for $e^+e^- \to 4$ jets [42]. I consider the three-jet cross section, where the jets are defined by the Durham jet algorithm [43]. The recombination prescription is given by the E scheme. I take the center-of-mass energy to be $\sqrt{Q^2} = m_Z$. The three-jet cross section is expanded as

$$\sigma_{3\text{-jet}} = \sigma_0\left[\frac{\alpha_s}{2\pi} A_{3\text{-jet}} + \left(\frac{\alpha_s}{2\pi}\right)^2 B_{3\text{-jet}} + \left(\frac{\alpha_s}{2\pi}\right)^3 C_{3\text{-jet}}\right],$$

where σ_0 is the LO cross section for $e^+e^- \to$ hadrons. The coefficients $A_{3\text{-jet}}$, $B_{3\text{-jet}}$ and $C_{3\text{-jet}}$ are given for the renormalization scale $\mu^2 = Q^2$ and various values of the jet defining parameter y_{cut} in Table I. The errors of $C_{3\text{-jet}}$ are from the Monte Carlo integration. For selected values of y_{cut} the contribution from the individual color factors to the NNLO coefficient $C_{3\text{-jet}}$ is shown in Table II. Finally, Fig. 2 shows the scale variation of the jet rate defined by

$$\frac{\sigma_{3\text{-jet}}}{\sigma_{\text{tot}}} = \frac{\alpha_s}{2\pi}\bar{A}_{3\text{-jet}} + \left(\frac{\alpha_s}{2\pi}\right)^2 \bar{B}_{3\text{-jet}} + \left(\frac{\alpha_s}{2\pi}\right)^3 \bar{C}_{3\text{-jet}},$$

where

$$\bar{A}_{3\text{-jet}} = A_{3\text{-jet}}, \qquad \bar{B}_{3\text{-jet}} = B_{3\text{-jet}} - A_{3\text{-jet}} A_{\text{tot}},$$
$$\bar{C}_{3\text{-jet}} = C_{3\text{-jet}} - B_{3\text{-jet}} A_{\text{tot}} - A_{3\text{-jet}}(B_{\text{tot}} - A_{\text{tot}}^2)$$

and $A_{\text{tot}} = 2$,

$$B_{\text{tot}} = \frac{N_c^2 - 1}{8N_c}\left[\left(\frac{243}{4} - 44\zeta_3\right)N_c + \frac{3}{4N_c} + (8\zeta_3 - 11)N_f\right].$$

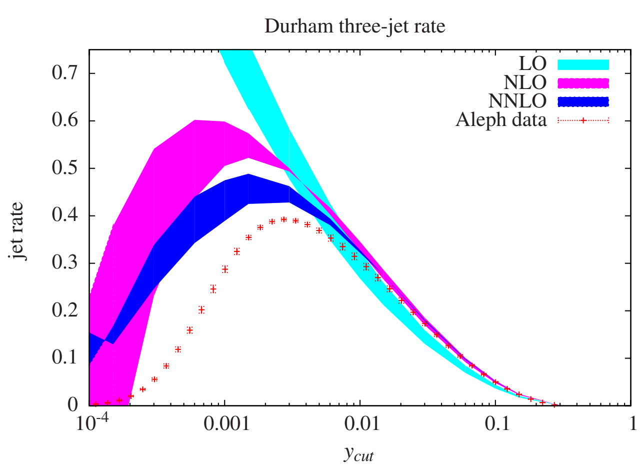

FIG. 2 (color online). The scale variation of the three-jet rate with the Durham jet algorithm at $\sqrt{Q^2} = m_Z$ with $\alpha_s(m_Z) = 0.118$. The bands give the range for the theoretical prediction obtained from varying the renormalization scale from $\mu = m_Z/2$ to $\mu = 2m_Z$.

TABLE I. The LO coefficient $A_{3\text{-jet}}$, the NLO coefficient $B_{3\text{-jet}}$ and the NNLO coefficient $C_{3\text{-jet}}$ for the three-jet cross section with the Durham jet algorithm and various values of y_{cut}.

y_{cut}	$A_{3\text{-jet}}$	$B_{3\text{-jet}}$	$C_{3\text{-jet}}$
0.3	0.02	0.13	-6 ± 3
0.1	2.12	34.3	$(2.0 \pm 0.2) \times 10^2$
0.03	7.63	113.8	$(6.7 \pm 0.6) \times 10^2$
0.01	15.7	152.6	$(-1.2 \pm 0.2) \times 10^3$
0.003	27.9	-6.5	$(-8.1 \pm 0.5) \times 10^3$
0.001	42.4	-562	$(-21 \pm 1) \times 10^3$
0.0003	61.8	-1.97×10^3	$(-25 \pm 3) \times 10^3$
0.0001	82.9	-4.36×10^3	$(7 \pm 5) \times 10^3$

TABLE II. The contributions from the individual color factors to the NNLO coefficient $C_{3\text{-jet}}$.

y_{cut}	N_c^2	$N_f N_c$
0.1	$(1.06 \pm 0.02) \times 10^3$	$(-9.80 \pm 0.06) \times 10^2$
0.01	$(4.6 \pm 0.2) \times 10^3$	$(-8.11 \pm 0.03) \times 10^3$
0.001	$(-29 \pm 1) \times 10^3$	$(-2.7 \pm 0.2) \times 10^3$
y_{cut}	N_c^0	N_f/N_c
0.1	-35 ± 1	21.9 ± 0.3
0.01	$(9.7 \pm 0.1) \times 10^2$	$(-2.66 \pm 0.02) \times 10^2$
0.001	$(7.09 \pm 0.08) \times 10^3$	$(-4.43 \pm 0.01) \times 10^3$
y_{cut}	N_c^{-2}	N_f^2
0.1	-0.49 ± 0.03	$(1.336 \pm 0.003) \times 10^2$
0.01	0.25 ± 0.15	$(1.646 \pm 0.002) \times 10^3$
0.001	$(3.38 \pm 0.01) \times 10^2$	$(7.41 \pm 0.01) \times 10^3$

The renormalization scale is varied from $\mu = m_Z/2$ to $\mu = 2m_Z$. In this plot the experimental measured values are also shown [44]. For values below $y_{cut} = 0.001$ the results of Ref. [8] differ significantly from the ones presented here.

Conclusions.—In this Letter I reported on the NNLO calculation for three-jet observables in electron-positron annihilation. Particular attention was paid to the cancellation of infrared singularities. I presented numerical results for the Durham three-jet cross section.

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