Number Theoretic Example of Scale-Free Topology Inducing Self-Organized Criticality

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In this Letter we present a general mechanism by which simple dynamics running on networks become self-organized critical for scale-free topologies. We illustrate this mechanism with a simple arithmetic model of division between integers, the division model. This is the simplest self-organized critical model advanced so far, and in this sense it may help to elucidate the mechanism of self-organization to criticality. Its simplicity allows analytical tractability, characterizing several scaling relations. Furthermore, its mathematical nature brings about interesting connections between statistical physics and number theoretical concepts. We show how this model can be understood as a self-organized stochastic process embedded on a network, where the onset of criticality is induced by the topology.

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In the late 1980s, Bak, Tang, and Wiesenfeld (BTW) [1,2] introduced the concept of Self-Organized Criticality (SOC) as a mechanism explaining how multicomponent systems can evolve naturally into barely stable self-organized critical structures without external "tuning" of parameters. This single contribution generated an enormous theoretical and experimental research interest in many areas of physics and interdisciplinary science, and many natural phenomena were claimed to exhibit SOC [3–5]. However, there was not a general accepted definition of what SOC exactly is, and the conditions under which it is expected to arise. In order to disengage the mechanism of self-organization to criticality one should likely focus on rather "simple" models, and in this sense Flyvbjerg recently introduced the "simplest SOC model" along with a workable definition of the phenomenon [6,7]; namely, "a driven, dissipative system consisting of a medium through which disturbances can propagate causing a modification of the medium, such that eventually, the disturbances are critical, and the medium is modified no more—in the statistical sense."

On the other hand, in the last years it has been realized that the dynamics of processes taking place on networks evidence a strong dependence on the network's topology [8,9]. Concretely, there exists a current interest on the possible relations between SOC behavior and scale-free networks [9], characterized by power law degree distributions $P(k) \sim k^{-\gamma}$, and how self-organized critical states can emerge when coupling topology and dynamics [10–13].

In this Letter, we introduce a rather simple and general mechanism by which the onset of criticality in the dynamics of self-organized systems is induced by the scale-free topology of the underlying network of interactions. To illustrate this mechanism we present a simple model, the division model from now on, based uniquely in the division between integers. We show that this model fulfills Flyvbjerg's definition of SOC and to our knowledge, constitutes the simplest SOC model advanced so far that is also analytically solvable. Interestingly, this model establishes connections between statistical physics and number theory (see [14] for a complete bibliography on this topic).

In number theory, a primitive set of N integers is the one for which none of the set elements divide exactly any other element [15–17]. Consider an ordered set of $M-1$ integers $\{2, 3, 4, \ldots, M\}$ (notice that zero and one are excluded, and that integers are not repeated), that we will name as the pool from now on. Suppose that we have extracted N elements from the pool to form a primitive set. The division model proceeds then by drawing integers at random from the remaining elements of the pool and introducing them in the set. Suppose that at time t the primitive set contains $N(t)$ elements. The algorithm updating rules are the following: (R1) Perturbation: an integer a is drawn from the pool at random and introduced in the primitive set. (R2) Dissipation: if a divides and/or is divided by say s elements of the primitive set, then we say that an instantaneous division avalanche of size s takes place, and these latter elements are returned to the pool, such that the set remains primitive but with a new size $N(t+1) = N(t) + 1 - s$.This process is then iterated, and we expect the primitive set to vary in size and composition accordingly. The system is *driven* and *dissipative* since integers are constantly introduced and removed from it, its size temporal evolution being characterized by $N(t)$.

In order to unveil the dynamics undergoing in the model, we have performed several Monte Carlo simulations for different values of the pool size M. In the top panel of Fig. 1 we have represented for illustration purposes a concrete realization of $N(t)$ for $M = 10^4$ and $N(0) = 0$. Note that after a transient, $N(t)$ self-organizes around an average stable value N_c, fluctuating around it. In the insert of the bottom panel of Fig. 1, we have plotted in log-log the power spectrum of $N(t)$: the system evidences $f^{-\beta}$ noise, with $\beta = 1.80 \pm 0.01$. The former fluctuations are indeed related to the fact that at each time step a new integer

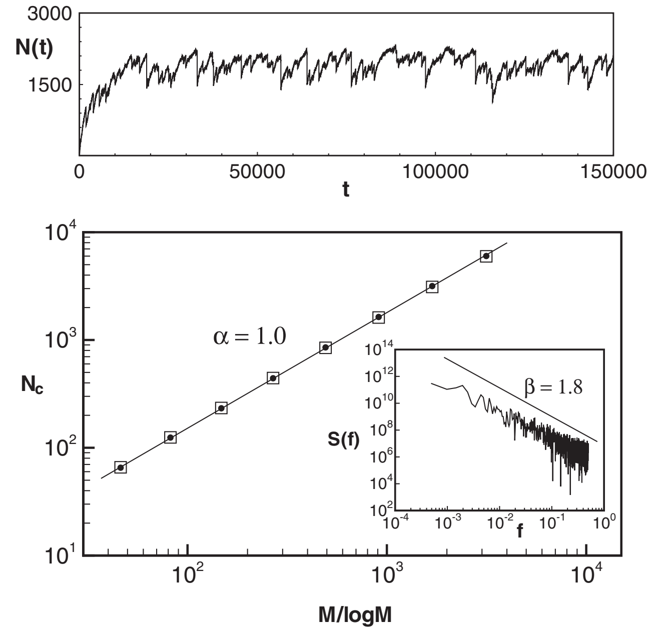

FIG. 1. Top: Single realization of the division model showing the time evolution of the primitive set size $N(t)$ for a pool size $M = 10^4$ and $N(0) = 0$. Notice that after a transient, $N(t)$ self-organizes around an average stable value N_c, fluctuating around it. Bottom: (black dots) Scaling behavior of the average stable value N_c as a function of the system's characteristic size $M/\log M$. The best fitting provides $N_c \sim (M/\log M)^\gamma$, with $\gamma = 1.05 \pm 0.01$. (squares) Scaling of N_c as predicted by Eq. (8). Inset: Plot in log-log of the power spectrum of $N(t)$, showing $f^{-\beta}$ noise with $\beta = 1.80 \pm 0.01$ (this latter value is the average of 10^5 realizations of $N(t)$ for 4096 time steps after the transient and $M = 10^4$).

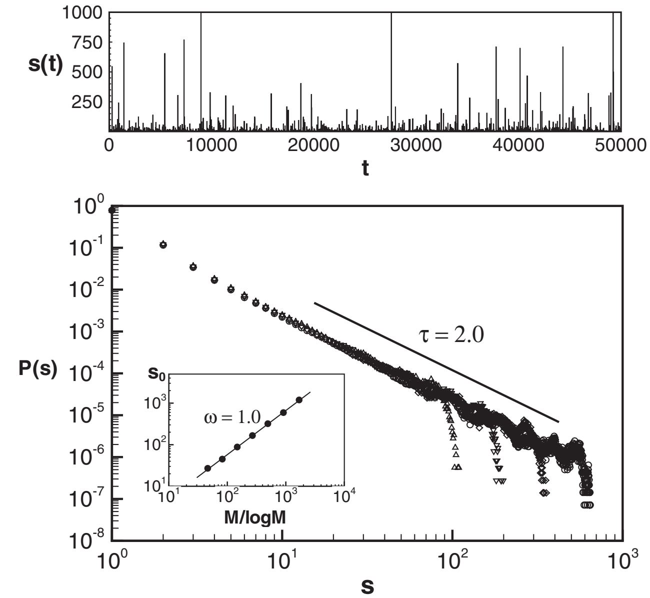

FIG. 2. Top: Single realization of the division model showing the time distribution of division avalanches. Bottom: Probability distribution $P(s)$ that a division avalanche of size s takes place in the system, for different pool sizes $M = 2^{10}$ (triangles), $M = 2^{11}$ (inverted triangles), $M = 2^{12}$ (diamonds) and $M = 2^{13}$ (circles). In every case we find $P(s) \sim s^{-\tau} \exp(s/s_0)$ with $\tau = 2.0 \pm 0.1$. Note that the power law relation evidences an exponential cutoff due to finite size effects at particular values of s_0. Inset: Scaling of the cutoff value s_0 as a function of the system's characteristic size $M/\log M$, with an exponent $\omega = 1.066 \pm 0.003$.

extracted from the pool enters the primitive set (external driving R1). Eventually (according to rule R2), a division avalanche can propagate and cause a *modification* in the size and composition of the primitive set. These avalanches constitute the *disturbances* of the system. In Fig. 2 (top) we have represented an example of the avalanche's size evolution in time. In the same figure (bottom) we show the probability $P(s)$ that a division avalanche of size s takes place, for different pool sizes M. These latter distributions are power laws $P(s) \sim s^{-\tau} \exp(s/s_0)$ with $\tau = 2.0 \pm 0.1$: disturbances are thus critical. Observe that the power law relation suffers from a crossover to exponential decay at a cutoff value s_0 due to finite size effects (pool is finite), and that the location of these cutoffs scales with the system's characteristic size $s_0 \sim (M/\log M)^\omega$ with $\omega = 1.066 \pm 0.003$, what is typically characteristic of a finite size critical state [3] (this characteristic size will be explained later in the text). We can conclude that according to Flyvbjerg's definition [6], the division model exhibits SOC. Division avalanches lead the system to different marginally stable states, that are nothing but primitive sets of different sizes and composition. Accordingly, for a given pool $[2, M]$, these time fluctuations generate a stochastic search in the configuration space of primitive sets.

In what follows we discuss analytical insights of the problem. Consider the *divisor function* [18] that provides the number of divisors of n, excluding integers 1 and n:

$$d(n) = \sum_{k=2}^{n-1}\left(\left\lfloor\frac{n}{k}\right\rfloor - \left\lfloor\frac{n-1}{k}\right\rfloor\right), \qquad (1)$$

where $\lfloor\cdots\rfloor$ stands for the integer part function. The average number of divisors of a given integer in the pool $[2, M]$ is then:

$$\frac{1}{M-1}\sum_{n=3}^{M} d(n) = \frac{1}{M-1}\sum_{k=2}^{M}\left\lfloor\frac{M}{k}\right\rfloor \simeq \sum_{k=2}^{M}\frac{1}{k} \simeq \log M + 2(\gamma - 1) + O\left(\frac{1}{\sqrt{M}}\right). \qquad (2)$$

Accordingly, the mean probability that two numbers a and b taken at random from $[2, M]$ are divisible is approximately $P = \Pr(a|b) + \Pr(b|a) \simeq 2\log M/M$. Moreover, if we assume that the N elements of the primitive set are uncorrelated, the probability that a new integer generates a division avalanche of size s is on average $(2\log M/M)N$. We can consequently build a mean field equation for the system's evolution, describing that at each time step an integer is introduced in the primitive set and a division avalanche of mean size $(2\log M/M)N$ takes place:

$$N(t+1) = N(t) + 1 - \left(\frac{2\log M}{M}\right) N(t), \tag{3}$$

whose fixed point $N_c = M/(2\log M)$, the stable value around which the system self-organizes, scales with the system's size as

$$N_c(M) \sim \frac{M}{\log M}. \tag{4}$$

Hitherto, we can conclude that the system's characteristic size is not M (pool size) as one should expect in the first place, but $M/\log M$. This scaling behavior has already been noticed in other number-theoretic models evidencing collective phenomena [19,20]. In Fig. 1 we have plotted (black dots) the values of N_c as a function of the characteristic size $M/\log M$ provided by Monte Carlo simulations of the model for different pool sizes $M = 2^8, 2^9, \ldots, 2^{15}$ (N_c has been estimated averaging $N(t)$ in the steady state). Note that the scaling relation (4) holds; however, the exact numerical values $N_c(M)$ are underestimated by Eq. (3). This is reasonable since we have assumed that the primitive set elements are uncorrelated, what is obviously not the case: observe for instance that any prime number $p \geq \lfloor M/2 \rfloor$ introduced in the primitive set will remain there forever. Fortunately this drawback of our mean field approximation can be improved by considering the function $D(n)$ that defines the exact number of divisors that a given integer $n \in [2, M]$ has, i.e., the amount of numbers in the pool that divide or are divided by n:

$$D(n) = d(n) + \left\lfloor \frac{M}{n} \right\rfloor - 1. \tag{5}$$

Define $p_n(t)$ as the probability that the integer n belongs at time t to the primitive set. Then, we have

$$p_n(t+1) = \left(1 - \frac{D(n)}{M - N(t)}\right) p_n(t) + \frac{1}{M - N(t)}[1 - p_n(t)], \tag{6}$$

that leads to a stationary survival probability in the primitive set:

$$p_n^* = \frac{1}{1 + D(n)}. \tag{7}$$

In Fig. 3 (right) we depict the stationary survival probability of integer n (black dots) obtained through numerical simulations for a system with $M = 50$, while squares represent the values of p_n^* as obtained from the Eq. (7). Note that there exists a remarkable agreement. We now can proceed to estimate the critical size values $N_c(M)$ as:

$$N_c(M) \approx \sum_{n=2}^{M} p_n^* = \sum_{n=2}^{M} \frac{1}{1 + D(n)}. \tag{8}$$

In Fig. 1 we have represented (squares) the values of $N_c(M)$ predicted by Eq. (8), showing good agreement with the numerics (black dots).

Finally, previous calculations point out that system's fluctuations, i.e., division-avalanches distribution $P(s)$ is proportional to the percentage of integers having s divisors. In order to test this conjecture, in Fig. 3 (left) we have plotted a histogram describing the amount of integers having a given number of divisors, obtained from computation of $D(n)$ for $M = 10^6$. The tail of this histogram follows a power law with exponent $\tau = 2.0$. This can be proven analytically as it follows: the numbers responsible for the tail of the preceding histogram are those that divide many others, i.e., rather small ones ($n \ll M$). A small number n divides typically $D(n) \simeq \lfloor \frac{M}{n} \rfloor$. Now, how many "small numbers" have $D(n)$ divisors? The answer is n, $n+1, \ldots, n+z$ where

$$\left\lfloor \frac{M}{n} \right\rfloor = \left\lfloor \frac{M}{n-1} \right\rfloor = \cdots = \left\lfloor \frac{M}{n-z} \right\rfloor. \tag{9}$$

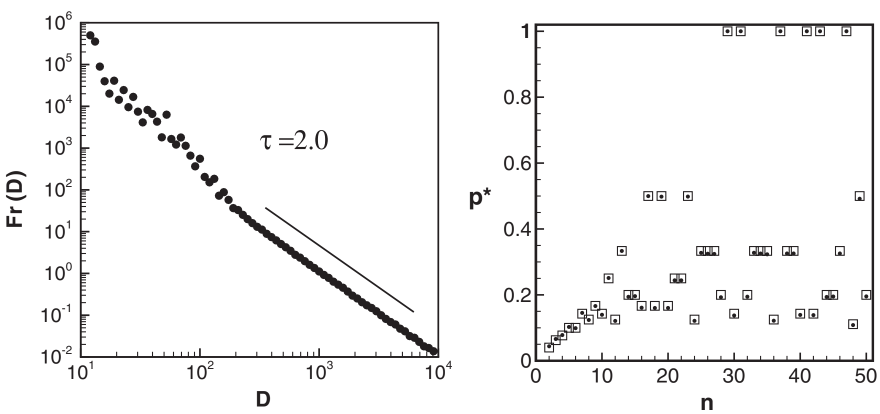

FIG. 3. Left: Histogram of the amount of integers in $[2, 10^6]$ that have D divisors. The histogram has been binned to reduce scatter. The best fitting provides a power law $P(D) \sim D^{-\tau}$ with $\tau = 2.01 \pm 0.01$, in agreement with $P(s)$ (see the text). Right: (black dots) Stationary survival probability of integer n in a primitive set for a pool size $M = 50$, obtained from Monte Carlo simulations of the model over 10^6 time steps (a preliminary transient of 10^4 time steps was discarded). (squares) Theoretical prediction of these survival probabilities according to Eq. (7).

The maximum value of z fulfills $\frac{M}{n-z} - \frac{M}{n} = 1$, that is $z \simeq n^2/M$. The frequency of $D(n)$ is thus $\text{fr}(D(n)) = n^2/M$, but since $s \equiv D(n) \simeq M/n$, we get $\text{fr}(s) \sim Ms^{-2}$, and finally normalizing, $P(s) \sim s^{-2}$.

Coming back to the Flyvbjerg's definition of SOC, which is the *medium* in the division model? Observe that the process can be understood as embedded in a network, where nodes are integers, and two nodes are linked if they are exactly divisible. The primitive set hence constitutes a subset of this network, that is dynamically modified according to the algorithm's rules. The degree of node n is $D(n)$, and consequently the degree distribution $P(k) \sim k^{-2}$ is scale-free. Hence the SOC behavior, which arises due to the divisibility properties of integers, can be understood as a sort of antipercolation process taking place in this scale-free network. Observe that the division model is a particular case of a more general class of self-organized models. These would be constituted by a network with M nodes having two possible states (*on/off*) where the following dynamics run: (R1) perturbation: at each time step a node in the state *off* is randomly chosen and switched *on*, (R2) dissipation: the s neighbors of the perturbed node that were in the state *on* in that time step are switched *off*, and we say that an instantaneous avalanche of size s has taken place. $N(t)$ measures the number of nodes in the state *on* as a function of time. Its evolution follows a mean field equation that generalizes Eq. (3):

$$N(t+1) = N(t) + 1 - \frac{\langle k \rangle}{M} N(t), \tag{10}$$

where $\langle k \rangle$ is the network's mean degree. Accordingly, in every case $N(t)$ will self-organize around an average value $N_c(M)$. Within regular or random networks, fluctuations (avalanches) around $N_c(M)$ will follow a Binomial or Poisson distribution, respectively. However, when the network is scale-free with degree distribution $P(k) \sim k^{-\gamma}$, fluctuations will follow a power law distribution $P(s) \sim s^{-\tau}$ with $\tau = \gamma$, and the dynamics will consequently be SOC. In this sense, we claim that scale-free topology induces criticality.

Some questions concerning this new mechanism can be depicted, namely: which is the relation between the specific topology of scale-free networks and the power spectra of the system's dynamics? Which physical or natural systems evidence this behavior?

With regard to the division model, the bridge between statistical physics and number theory should also be investigated in depth. This includes possible generalizations of this model to other related sets such as k-primitive sets [21], where every number divides or is divided by at least k others (k acting as a threshold parameter), to relatively primitive sets [22] and to cross-primitive sets [16] (where this will introduce coupled SOC models). From the computational viewpoint [23], properties of the model as a primitive set generator should also be studied. Of special interest is the task of determining the maximal size of a k-primitive set [16,21], something that can be studied within the division model through extreme value theory [4].

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