Phase Reduction of Stochastic Limit Cycle Oscillators

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We point out that for an oscillator subjected to noise the conventional phase equation is not a proper approximation even for weak noise. We present a phase reduction method valid for an oscillator subjected to weak white Gaussian noise. Numerical evidence demonstrates that the phase equation properly approximates dynamics of the original oscillator. Moreover, we show that, in general, noise causes a shift of the oscillator frequency and discuss its effects on entrainment.

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Many physical systems can be mathematically modeled by limit cycle oscillators. It is well known that oscillator systems can exhibit a variety of behaviors. A fundamental theoretical technique for studying oscillator dynamics is the phase reduction method, which allows one to approximately describe the oscillator dynamics by a simpler equation for the phase variable only (e.g., [1]). This method has been widely and successfully applied to coupled oscillators or an oscillator subjected to an external periodic signal.

Dynamics of oscillators subjected to noises has also attracted much interest (e.g., [1]) and it has been studied by using the phase reduction method in a number of works [2–5]. It has been believed that the phase reduction method gives a good approximation for any type of weak signal including noises [1]. Therefore, this method has been also applied in a conventional way to the stochastic differential equations which describe oscillators subjected to white Gaussian noises in [2–5]. However, as we will show, the phase equation obtained in such a way is incorrect in the sense that it does not properly approximate the dynamics of the original full oscillator system even in the weak noise limit.

Phase reduction is a powerful method for studying essential dynamics of oscillators. It is essential to extend its domain of applicability to the case of oscillators subjected to noises. We derive a phase equation valid for oscillators subjected to white Gaussian noises. It is numerically demonstrated that the present phase equation properly approximates dynamics of the original full oscillator system while the conventional phase equation fails. The present phase equation reveals that in general a noise causes a shift of the oscillator frequency. This effect cannot be described by the conventional phase equation. We show that the noise-induced frequency shift (NIFS) causes significant influences on entrainment of oscillators.

Let $\boldsymbol{X} = (x_1, \ldots, x_N) \in \mathbb{R}^N$ be a state variable vector and consider the stochastic differential equation

$$\dot{\boldsymbol{X}} = \boldsymbol{F}(\boldsymbol{X}) + \boldsymbol{G}(\boldsymbol{X})\xi(t), \tag{1}$$

where $\boldsymbol{F}$ is an unperturbed vector field, $\boldsymbol{G}$ is a vector function, and $\xi(t)$ is the white Gaussian noise such that $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(s) \rangle = 2D\delta(t-s)$, where $\langle \cdots \rangle$ denotes averaging over the realizations of ξ and δ is Dirac's delta function. We call the constant $D > 0$ the noise intensity. The unperturbed system $\dot{\boldsymbol{X}} = \boldsymbol{F}(\boldsymbol{X})$ is assumed to have a limit cycle with a frequency ω. We employ the Stratonovich interpretation for Eq. (1). In this interpretation, ordinary variable transformation in differential equation can be applied.

Consider the unperturbed system $\dot{\boldsymbol{X}} = \boldsymbol{F}(\boldsymbol{X})$ and let $\boldsymbol{X}_0(t)$ be its limit cycle solution. A phase coordinate ϕ can be defined in a neighborhood U of $\boldsymbol{X}_0$ in phase space. We define ϕ so that $(\mathrm{grad}_{\boldsymbol{X}}\phi) \cdot \boldsymbol{F}(\boldsymbol{X}) = \omega$ may hold for any points in U. We can define the other $N-1$ coordinates $\boldsymbol{r} = (r_1, \ldots, r_{N-1})$ in U. We assume that $\boldsymbol{r} = \boldsymbol{a}$ on the limit cycle, where $\boldsymbol{a} = (a_1, \ldots, a_{N-1})$ is a constant vector. If we perform the transformation $(x_1, \ldots, x_N) \mapsto (\phi, r_1, \ldots, r_{N-1})$ in Eq. (1), we have

$$\dot{\phi} = \omega + h(\phi, \boldsymbol{r})\xi(t), \tag{2}$$

$$\dot{r}_i = f_i(\phi, \boldsymbol{r}) + g_i(\phi, \boldsymbol{r})\xi(t), \tag{3}$$

where $i = 1, \ldots, N-1$. The functions h, f_i, and g_i are defined as follows: $h(\phi, \boldsymbol{r}) = (\mathrm{grad}_{\boldsymbol{X}}\phi) \cdot \boldsymbol{G}(\boldsymbol{X}(\phi, \boldsymbol{r}))$, $f_i(\phi, \boldsymbol{r}) = (\mathrm{grad}_{\boldsymbol{X}} r_i) \cdot \boldsymbol{F}(\boldsymbol{X}(\phi, \boldsymbol{r}))$, $g_i(\phi, \boldsymbol{r}) = (\mathrm{grad}_{\boldsymbol{X}} r_i) \cdot \boldsymbol{G}(\boldsymbol{X}(\phi, \boldsymbol{r}))$, where the gradients are evaluated at the point $\boldsymbol{X}(\phi, \boldsymbol{r})$. They are 2π-periodic functions of ϕ.

Stratonovich stochastic differential equations (2) and (3) can be converted into the equivalent Ito stochastic differential equations [6]. The ϕ component of this Ito-type equation is obtained as follows:

$$\dot{\phi} = \omega + D\left[\frac{\partial h(\phi, \boldsymbol{r})}{\partial \phi} h(\phi, \boldsymbol{r}) + \sum_{i=1}^{N-1} \frac{\partial h(\phi, \boldsymbol{r})}{\partial r_i} g_i(\phi, \boldsymbol{r})\right] + h(\phi, \boldsymbol{r})\xi(t). \tag{4}$$

In the case of weak noise $0 < D \ll 1$, the deviation of $\boldsymbol{r}$ from $\boldsymbol{a}$ is expected to be small. Thus, we can use the approximation $\boldsymbol{r} = \boldsymbol{a}$ in Eq. (4) and arrive at

$$\dot{\phi} = \omega + D[Z(\phi)Z'(\phi) + Y(\phi)] + Z(\phi)\xi(t), \tag{5}$$

where $Z(\phi)$ and $Y(\phi)$ are given by

$$Z(\phi) = h(\phi, \boldsymbol{a}), \qquad Y(\phi) = \sum_{i=1}^{N-1} \frac{\partial h(\phi, \boldsymbol{a})}{\partial r_i} g_i(\phi, \boldsymbol{a}). \quad (6)$$

Since h and g_i are 2π periodic, $Z(\phi + 2\pi) = Z(\phi)$ and $Y(\phi + 2\pi) = Y(\phi)$ hold. The Ito-type phase equation for the noise-driven oscillator (1) is given by Eq. (5).

The conventional procedure of phase reduction consists in substituting $\boldsymbol{r} = \boldsymbol{a}$ in Eq. (2). In the previous studies [2–5], the authors obtained the Stratonovich-type equation $\dot{\phi} = \omega + Z(\phi)\xi(t)$, which is of the standard form of phase equation, according to this procedure. The equivalent Ito-type phase equation is given by

$$\dot{\phi} = \omega + DZ(\phi)Z'(\phi) + Z(\phi)\xi(t). \quad (7)$$

Comparison of Eqs. (5) and (7) clearly shows that the term $DY(\phi)$ is dropped in the previously used equation (7). This term is $O(D)$ and is of the same order as $DZ(\phi)Z'(\phi)$. Thus, in general, Eq. (7) does not correctly describe the original oscillator dynamics even in the lowest order approximation. It should be noted that the approximation $\boldsymbol{r} = \boldsymbol{a}$ has to be performed in the Ito-type equation for ϕ, but not in the Stratonovich-type one, to obtain the correct phase equation since the term $DY(\phi)$ due to correlations between fluctuations in $\boldsymbol{r}$ and ξ has to be included.

We outline a proof of Eq. (5). For simplicity, we consider the case $N = 2$ and denote r_i, f_i, and g_i in Eq. (3) by r, f, and g, respectively. Generalization for larger N is straightforward. We can assume without loss of generality that $r = 0$ on the limit cycle. The Fokker-Planck equation for Eqs. (2) and (3) is given by

$$\begin{aligned}\frac{\partial Q}{\partial t} = &-\frac{\partial}{\partial \phi}[\{\omega + D(h_\phi h + h_r g)\}Q] + D\frac{\partial^2[h^2 Q]}{\partial \phi^2} \\ &- \frac{\partial}{\partial r}[\{f + D(g_\phi h + g_r g)\}Q] + 2D\frac{\partial^2[hgQ]}{\partial \phi \partial r} \\ &+ D\frac{\partial^2[g^2 Q]}{\partial r^2},\end{aligned} \quad (8)$$

where $Q(t, \phi, r)$ is the time-dependent probability distribution and the subscripts ϕ and r stand for partial derivatives with respect to ϕ and r, respectively.

Let ρ be a constant such that the region $\{(\phi, r); -\rho \leq r \leq \rho\}$ is in the neighborhood U. When $D = 0$, the steady distribution is given by $Q_0(\phi, r) = (2\pi)^{-1}\delta(r)$, where δ is Dirac's delta function. For small $D > 0$, the steady distribution Q_0 still localizes near $r = 0$ and it rapidly decreases with increasing $|r|$ because of asymptotic stability of the limit cycle. Assume that t is sufficiently large. Then, this property also holds for $Q(t, \phi, r)$, since Q converges to Q_0. Thus, $Q = 0$ and $\partial Q/\partial r = 0$ approximately hold at $r = \pm\rho$ for small D.

We introduce the marginal distribution $P(t, \phi) \equiv \int_{-\rho}^{\rho} Q(t, \phi, r)dr$, neglecting a small probability over the region $|r| > \rho$. We integrate Eq. (8) with respect to r over the interval $I = [-\rho, \rho]$ to obtain an approximate Fokker-Planck equation for P. The last three terms in Eq. (8), which include the derivative $\partial/\partial r$, vanish after the integration due to the two conditions $Q = 0$ and $\partial Q/\partial r = 0$ at $r = \pm\rho$: for example, $\int_I \frac{\partial}{\partial r}[\{f + D(g_\phi h + g_r g)\}Q]dr = [\{f + D(g_\phi h + g_r g)\}Q]_{-\rho}^{\rho} = 0$. Therefore, after integrating Eq. (8), we have

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial \phi}\int_I (\omega + DK_1)Qdr + D\frac{\partial^2}{\partial \phi^2}\int_I K_2 Qdr, \quad (9)$$

where K_1 and K_2 are functions of ϕ and r given by $K_1 = h_\phi h + h_r g$ and $K_2 = h^2$.

The functions K_1 and K_2 can be expanded in the forms $K_1 = h_\phi(\phi, 0)h(\phi, 0) + h_r(\phi, 0)g(\phi, 0) + rR_1(\phi, r)$ and $K_2 = h(\phi, 0)^2 + rR_2(\phi, r)$, where R_1 and R_2 are functions of $O(1)$ with respect to r. Since $Z(\phi) = h(\phi, 0)$ and $Y(\phi) = h_r(\phi, 0)g(\phi, 0)$ from Eq. (6), K_1 and K_2 are rewritten as $K_1 = Z(\phi)Z'(\phi) + Y(\phi) + rR_1(\phi, r)$ and $K_2 = Z(\phi)^2 + rR_2(\phi, r)$. Consider the integrals $\int_I rR_i Qdr$, $i = 1, 2$. Recall that the steady distribution Q_0 of Eq. (8) satisfies $\lim_{D\to 0} Q_0(\phi, r) = (2\pi)^{-1}\delta(r)$. Since $Q(t, \phi, r) \simeq Q_0(\phi, r)$ holds, the profile of $Q(t, \phi, r)$ in r may be approximated by $\delta(r)$ in the limit $D \to 0$. If we use this approximation and note that $rR_i = O(r)$, we have $\lim_{D\to 0}\int_I rR_i Qdr = 0$. This implies that $D\frac{\partial}{\partial \phi} \times \int_I rR_1 Qdr = o(D)$ and $D\frac{\partial^2}{\partial \phi^2}\int_I rR_2 Qdr = o(D)$. If we substitute the expansions of K_1 and K_2 into Eq. (9) and use these facts, we can obtain the approximate Fokker-Planck equation up to $O(D)$ as follows:

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial \phi}[\{\omega + D(ZZ' + Y)\}P] + D\frac{\partial^2}{\partial \phi^2}[Z^2 P]. \quad (10)$$

The stochastic differential equation equivalent to Eq. (10) is given by Eq. (5). Thus, we may conclude that Eq. (5) is a proper phase equation for system (1).

We calculate the steady probability distribution $P_0(\phi)$ of the phase variable and the mean frequency Ω. We will compare these quantities between a two dimensional oscillator model and its reduced phase model.

We consider Eq. (10) with the boundary condition $P(t, 0) = P(t, 2\pi)$, which is equivalent to Eq. (5). The steady solution $P_0(\phi)$ is obtained by assuming $\partial P/\partial t = 0$ in Eq. (10). If we construct an asymptotic solution for P_0 in the power of $\varepsilon \equiv D/\omega$, then up to $O(\varepsilon)$ we obtain

$$P_0(\phi) = \frac{1}{2\pi} + \frac{\varepsilon}{2\pi}[Z(\phi)Z'(\phi) - Y(\phi) + \bar{Y}] + o(\varepsilon), \quad (11)$$

where $\bar{Y}$ is defined by $\bar{Y} = (2\pi)^{-1}\int_0^{2\pi} Y(\phi)d\phi$.

The mean frequency Ω of the oscillator is defined by $\Omega = \lim_{T\to\infty} T^{-1}\int_0^T \dot{\phi}(t)dt$. This can be calculated by replacing the time average with the ensemble average; i.e., $\Omega = \langle\dot{\phi}\rangle$. There is no correlation between ϕ and ξ in the Ito equation. If we take the ensemble average of Eq. (5), we have $\Omega = \omega + D\langle Z(\phi)Z'(\phi) + Y(\phi)\rangle$, where we used the fact $\langle Z(\phi)\xi(t)\rangle = \langle Z(\phi)\rangle\langle \xi(t)\rangle = 0$. For an arbitrary function $A(\phi)$, its ensemble average can be calculated by using P_0; i.e., $\langle A\rangle = \int_0^{2\pi} A(\phi)P_0(\phi)d\phi$. If we use Eq. (11), we obtain Ω up to $O(\varepsilon)$ as follows:

$$\Omega/\omega = 1 + \varepsilon\bar{Y} + o(\varepsilon). \qquad (12)$$

Since the white Gaussian noise has no characteristic frequency, intuitively, one might expect that it causes no change in the oscillator frequency. However, this is not the case. Equation (12) shows that a white Gaussian noise does change Ω; i.e., the NIFS occurs. It depends on the sign of $\bar{Y}$ whether Ω increases or decreases as the noise intensity increases.

Equations (11) and (12) show that the term $Y(\phi)$ in Eq. (5) significantly affects both $P_0(\phi)$ and Ω in the first order of ε. In particular, as shown by Eq. (12), the first-order frequency shift is determined only from $Y(\phi)$. Therefore, it is crucially important to include the term $Y(\phi)$ into the phase equation as in Eq. (5). It is clear that the previously used phase equation (7) cannot give proper approximations for $P_0(\phi)$ and Ω.

In order to validate Eq. (5), we compare $P_0(\phi)$ and Ω between theoretical and numerical results. As an example, we use the Stuart-Landau (SL) oscillator [1]: $\boldsymbol{X} = (x, y)$ and $\boldsymbol{F}(\boldsymbol{X}) = (x - c_0 y - (x^2 + y^2)(x - c_2 y), c_0 x + y - (x^2 + y^2)(c_2 x + y))$ in Eq. (1), where c_0 and c_2 are constants. The noise-free SL oscillator has the limit cycle $\boldsymbol{X}_0(t) = (\cos\omega t, \sin\omega t)$, where the natural frequency ω is given by $\omega = c_0 - c_2$. If we define the coordinates (ϕ, r) by $x = r\cos(\phi + c_2 \ln r)$ and $y = r\sin(\phi + c_2 \ln r)$, then ϕ gives the phase variable and the limit cycle is represented by $r = 1$.

We use the two types of $\boldsymbol{G}$: $\boldsymbol{G}_1 = (1, 0)$ and $\boldsymbol{G}_2 = (x, 0)$. For $\boldsymbol{G}_1$, $Z(\phi)$ and $Y(\phi)$ are given by $Z(\phi) = -(\sin\phi + c_2\cos\phi)$ and $Y(\phi) = \{(1 + c_2^2)/2\}\sin 2\phi$. For $\boldsymbol{G}_2$, they are $Z(\phi) = -\cos\phi(\sin\phi + c_2\cos\phi)$ and $Y(\phi) = c_2\cos^2\phi(-\cos 2\phi + c_2\sin 2\phi)$. Approximations for $P_0(\phi)$ and Ω can be obtained by substituting these expressions for $Z(\phi)$ and $Y(\phi)$ into Eqs. (11) and (12).

In Figs. 1(a)–1(d), numerical and theoretical results for $P_0(\phi)$ are compared: the filled circle and solid line represent $P_0(\phi)$ obtained by numerically solving the equation of the SL oscillator and that given by Eq. (11), respectively. Theoretical predictions made by Eq. (7), which are obtained just by setting $Y = 0$ in Eq. (11), are also shown by the dashed line. Figures 1(a) and 1(b) are for $\boldsymbol{G}_1$ while Figs. 1(c) and 1(d) are for $\boldsymbol{G}_2$. It is clear that the present phase model (5) gives precise approximations in all the cases. The agreements are excellent. In contrast, the previously used phase model (7) does not give proper approximations at all in spite of the weak noise intensity.

Figures 2(a) and 2(b) show the mean frequency Ω plotted against $\varepsilon = D/\omega$ for $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$, respectively, where the natural frequency is set as $\omega = 1$. The numerical results obtained by solving the equation of the SL oscillator

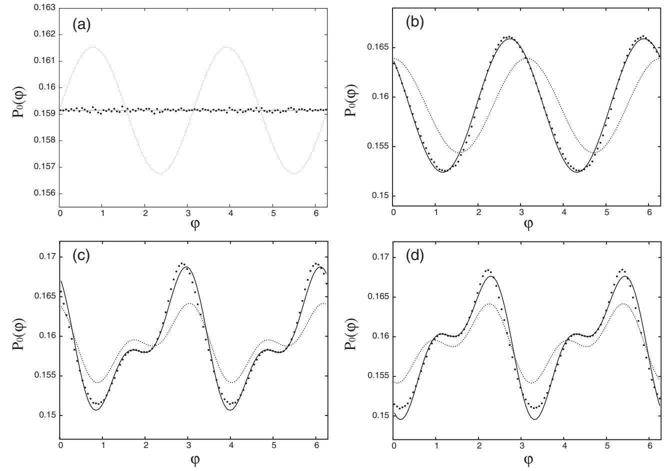

FIG. 1. Steady probability distribution $P_0(\phi)$ of phase for noise-driven SL oscillator. Numerical result (●), analytical result Eq. (11) (solid line), and that obtained from Eq. (7) (dashed line) are shown for $\varepsilon = 0.03$. (a) $\boldsymbol{G}_1$ and $(c_0, c_2) = (1, 0)$, (b) $\boldsymbol{G}_1$ and $(c_0, c_2) = (2, 1)$, (c) $\boldsymbol{G}_2$ and $(c_0, c_2) = (2, 1)$, (d) $\boldsymbol{G}_2$ and $(c_0, c_2) = (0, -1)$.

are shown by filled or open circles. The theoretical estimations given by Eq. (12) are shown by a solid line or a dashed line. The theoretical estimation is $\Omega/\omega = 1 + o(\varepsilon)$ for $\boldsymbol{G}_1$, which is constant up to $O(\varepsilon)$. In Fig. 2(a), the numerically obtained Ω is almost constant for $(c_0, c_2) = (1, 0)$. This coincides with the above theoretical estimation. For $(c_0, c_2) = (2, 1)$, the numerically obtained Ω is not constant but increases with increasing ε. However, this increase is not linear with respect to ε but a higher order one as shown in the inset. In this sense, an agreement between the numerical and theoretical results is confirmed up to $O(\varepsilon)$. In the case of $\boldsymbol{G}_2$, the theoretical estimation is given by $\Omega/\omega = 1 - (c_2/4)\varepsilon + o(\varepsilon)$, which has a nonvanishing term of $O(\varepsilon)$ except for $c_2 = 0$. This indicates that Ω can either increase or decrease, depending on the sign of c_2. In Fig. 2(b), this estimation well agrees with the numerical result in each of the cases $(c_0, c_2) = (2, 1)$ and $(0, -1)$. If we use Eq. (7) instead of Eq. (5), then we obtain

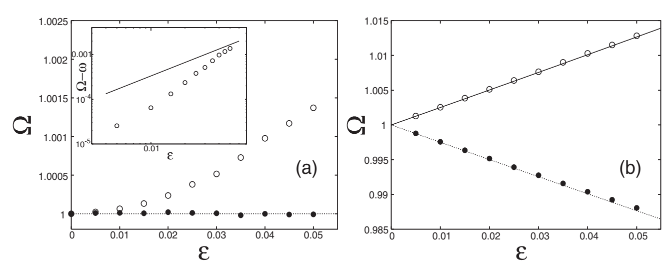

FIG. 2. Mean frequency Ω vs ε for (a) $\boldsymbol{G}_1$ and (b) $\boldsymbol{G}_2$. Numerical result (symbol) and analytical result Eq. (12) (line) are shown. (a) $(c_0, c_2) = (1, 0)$ (●, dashed line) and $(c_0, c_2) = (2, 1)$ (○, dashed line), (b) $(c_0, c_2) = (2, 1)$ (●, dashed line) and $(c_0, c_2) = (0, -1)$ (○, solid line). The inset in (a) is logarithmic plot of $\Omega - \omega$ vs ε for $(c_0, c_2) = (2, 1)$, where reference line for the scaling law ε^1 is also shown.

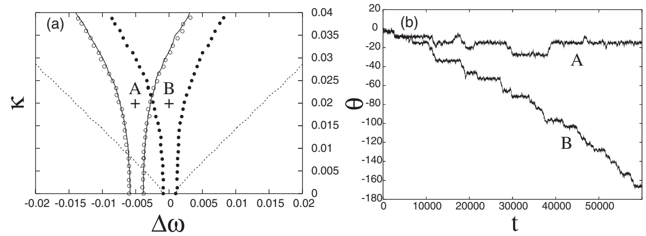

FIG. 3. (a) Frequency locking region of SL oscillator with periodic signal, where $(c_0, c_2) = (2, 1)$. Locking regions for $D = 0$ (dashed line) and 0.02 (solid line) are shown. Regions for $D = 0.02$ obtained by Eqs. (5) (○) and (7) (●) with the periodic term are also shown. (b) θ vs t for cases A and B ($D = 0.02$).

the estimation $\Omega/\omega = 1 + o(\varepsilon)$ for $\boldsymbol{G}_2$, in which the $O(\varepsilon)$ term vanishes. This estimation apparently disagrees with the numerical results.

Figures 1 and 2 clearly demonstrate that Eq. (5) precisely approximates dynamics of oscillators with weak white Gaussian noises. In addition, it is apparent that the previously used equation (7) is erroneous.

The present phase equation (5) has revealed that a noise shifts the frequency of an oscillator. We show that this NIFS plays an important role in entrainment phenomena. As a simple example for demonstration, let us consider the SL oscillator with a noise and a periodic signal, which is described by $\dot{\boldsymbol{X}} = \boldsymbol{F}(\boldsymbol{X}) + \boldsymbol{G}_2(\boldsymbol{X})\xi(t) + \boldsymbol{K}\cos(\omega_0 t)$, where $\boldsymbol{X} = (x, y)$, $\boldsymbol{F}$ is the vector field of the SL oscillator with $(c_0, c_2) = (2, 1)$, $\boldsymbol{G}_2 = (x, 0)$, $\xi(t)$ is the white Gaussian noise, and $\boldsymbol{K} = (\kappa, 0)$ is a constant vector. We define the detuning $\Delta\omega$ by $\Delta\omega = \omega_0 - \omega$, where ω is the natural frequency.

Figure 3(a) shows frequency locking regions in the $(\Delta\omega, \kappa)$ plane for $D = 0$ and 0.02. The locking condition $|\Omega - \omega_0| < 10^{-3}$ holds in a wedge-shaped region between two boundaries shown by a dashed line or a solid line. The locking region is centered at $\Delta\omega = 0$ when $D = 0$. In contrast, when $D = 0.02$, the center of the locking region clearly shifts to the negative direction as if the oscillator has a smaller natural frequency. Locking regions obtained by numerically solving the corresponding phase equations, which are obtained by adding the periodic term $(\text{grad}_X \phi) \cdot \boldsymbol{K}\cos(\omega_0 t)$ to Eq. (5) and (7), are also shown. Equation (5) with the periodic term properly describes this effect, showing a good agreement, while Eq. (7) with the periodic term does not, which lacks the term $Y(\phi)$ and cannot describe the NIFS. The amount of the center-frequency shift in $\Delta\omega$ is -0.005, and this value coincides with the amount of NIFS obtained from the analytical formula $\Omega/\omega = 1 - (c_2/4)\varepsilon$ with $\omega = 1$, $c_2 = 1$, and $\varepsilon = 0.02$. Thus, we may conclude that the center-frequency shift of the locking region is an effect due to the NIFS. In Fig. 3(b), the phase difference $\theta = \phi - \omega_0 t$ is plotted against time t for two cases indicated in Fig. 3(a). The average laminar time is much longer for A than B and a better-quality locking is achieved in case A, although case B has a smaller original detuning and a better-quality locking is expected from Eq. (7).

The above results clearly indicate that the original detuning $\omega_0 - \omega$ is not relevant, but the effective detuning $\omega_0 - \Omega$, where Ω is given by Eq. (12), is an important parameter, which characterizes the nature of entrainment in an oscillator with noise. It is expected that the effective detuning, which takes into account the NIFS, is also important for characterizing the entrainment transition in mutually coupled two oscillators or an ensemble of many oscillators when they are subjected to noises. Theoretical studies lacking the NIFS effect could lead to incorrect scenarios for entrainment in noisy oscillator systems. Suppose an ensemble of many nonidentical noisy oscillators: the amount of NIFS of each oscillator is different. It could happen that the natural-frequency and effective-frequency distributions are qualitatively different; for example, the latter could have a double-peak profile while the former could have a single-peak profile. The conventional theory lacking the NIFS effect describes the entrainment transition scenario based on the natural-frequency distribution. However, the transition is expected to be dominated by the effective-frequency distribution. Then, the conventional theory could lead to an incorrect scenario. We emphasize that it is essential to use Eq. (5) to correctly describe and understand the nature of entrainment in various physical systems subjected to noises.

In conclusion, we have developed the phase reduction method valid for oscillators subjected to weak white Gaussian noises. We showed that in general the NIFS occurs and discussed its effects on entrainment phenomenon. The present results suggest that a modification of the phase reduction method is necessary also for noises other than the white Gaussian noise, which have finite correlation times.

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