Dynamics of Matter-Wave Solitons in a Ratchet Potential

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We study the dynamics of bright solitons formed in a Bose-Einstein condensate with attractive atomic interactions perturbed by a weak bichromatic optical lattice potential. The lattice depth is a biperiodic function of time with a zero mean, which realizes a flashing ratchet for matter-wave solitons. We find that the average velocity of a soliton and the soliton current induced by the ratchet depend on the number of atoms in the soliton. As a consequence, soliton transport can be induced through scattering of different solitons. In the regime when matter-wave solitons are narrow compared to the lattice period the dynamics is well described by the effective Hamiltonian theory.

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The ratchet effect, i.e., rectified average current induced by an asymmetric potential and unbiased zero-mean driving, has historically attracted a lot of attention due to its possible relevance to biological transport and molecular motors, and prospects for nanotechnology [1]. Both in classical and quantum systems, the ratchet effect has been studied in dissipative as well as Hamiltonian regimes [2,3] and is associated with a broken space-time symmetry of the perturbing potential [4]. Directed ratchet transport can be experimentally implemented in different physical systems ranging from semiconductor heterostructures to quantum dots, Josephson junctions, and cold atoms in optical lattices [5]. Recently, the interest in optical ratchets and especially in the effect of interaction on the ratchet transport [6] has resurged with the experimental advances in implementing atomic ratchets for Bose-Einstein condensates (BECs) [7].

A BEC of atoms with negative scattering length supports the existence of localized collective excitations, bright matter-wave solitons. It is therefore interesting to explore the effect of ratchet potentials on the transport and interaction properties of such particlelike excitations. It is especially important because theoretical studies of the soliton ratchets so far were mostly focused on topological solitons [8], whereas a BEC is a perfect test-bed for the study of the ratchet dynamics of a general class of nontopological solitons governed by a continuous Gross-Pitaevskii (nonlinear Schrödinger) equation.

In this Letter we study, for the first time to our knowledge, the effect of the ratchet potential on nontopological nondissipative bright matter-wave solitons. The ratchet potential for matter waves can be realized by means of a bichromatic optical lattice which is "flashed" on and off in such a way that its time-averaged amplitude vanishes. We show that both the ratchet effect and soliton directed transport are observed in such a system even in the absence of losses, which sets it apart from previously studied dissipative nonlinear systems subjected to ratchet potentials [9]. A weak potential does not affect the soliton shape, especially when the soliton is strongly localized, which justifies its treatment as an effective classical particle. However, we show that the ratchet effect is even more pronounced when the extended nature of the excitation cannot be ignored and the soliton width exceeds the period of the lattice. Furthermore, we investigate the influence of the ratchet on soliton scattering, and show that multiple collisions between solitons may provide a space averaging mechanism that can enable observation of a soliton current in a ratchet potential.

We consider a matter-wave soliton formed in a strongly elongated condensate cloud [10] subjected to a flashing one-dimensional (1D) optical lattice. As long as the energy of the longitudinal excitations is not sufficient to excite the transverse modes of the BEC, the system can be described by the 1D Gross-Pitaevskii (GP) equation:

$$i\frac{\partial \Psi}{\partial t} + \frac{1}{2}\frac{\partial^2 \Psi}{\partial x^2} + |\Psi|^2\Psi - V(x,t)\Psi = 0, \qquad (1)$$

where the Fourier-synthesized lattice potential,

$$V(x,t) = V_0 f(t)[\cos(x) + \cos(2x + \phi)], \qquad (2)$$

is driven biperiodically: $f(t) = \sin(\omega t) + \sin(2\omega t)$, and V_0 depends on the intensity of the laser beams forming the lattice. Quantum transport of ultracold atoms in the stationary potential of the form (2) has been recently studied experimentally [11]. The choice of the potential (2) and the time dependence $f(t)$ ensures that both the space- and time-inversion symmetries are broken if $\phi \neq 0, \pi$. According to the symmetry analysis [4], this can allow a directed transport of matter-wave solitons.

In the model (1) the energy, length, and frequency are measured in the units of $E_0 = \hbar^2 k^2/m$, $a_0 = 1/k$, and

$\omega_0 = \hbar k^2/m$, respectively, where m is the atomic mass, and k is the wave vector of the optical lattice. The 3D model was reduced to the 1D GP equation by assuming that the wave function is separable, $\psi_{3D}(x, y, z) = \psi_{1D}(x)\Phi(y, z)$, where $\Phi(y, z)$ is the normalized ground state wave function of the 2D harmonic trap with the trapping frequency $\omega_\perp$. The wave function in Eq. (1) therefore relates to ψ_{1D} as follows: $\Psi = \psi_{1D}\sqrt{g_{1D}}$, where $g_{1D} = 2(a_s\omega_\perp)/(a_0\omega_0)$ is the renormalized interaction coefficient that characterizes the s-wave interaction of atoms with a scattering length a_s. The number of atoms in the system is given by $\mathcal{N} = N/g_{1D}$, where $N = \int|\Psi|^2dx$ is the norm of the dimensionless wave function.

The relationship between the dimensionless and physical parameters of the system depends on the particular experimental conditions. Here we consider the experimental setup of Ref. [10], where a bright soliton forms in the ^{7}Li cloud with a modified scattering length $a_s \approx -0.21$ nm trapped in a quasi-one-dimensional atomic waveguide with $\omega_\perp = 2\pi \times 710$ Hz. Furthermore, we consider a flashing optical lattice applied in the direction of the waveguide and formed by CO_2 laser beams with a wavelength $\lambda = 10.62$ μm crossed at the angle $\theta = 38°$. Given these physical parameters, our scaling units of length and frequency take the values $a_0 = \lambda/[4\pi\sin(\theta/2)] = 2.52$ μm and $\omega_0 = 2\pi \times 224$ Hz. A stable bright soliton typically created in the experiment [10] contains $\mathcal{N} \approx 5 \times 10^3$ atoms, which corresponds to $N \approx 2.62$. We note that by changing the angle θ it is possible to achieve smaller or larger values of a_0, and hence of N, for the same number of atoms, $\mathcal{N}$.

The Hamiltonian description of the mean field [12] allows us to write an effective energy integral associated with the equation of motion (1):

$$H(x,t) = \int_{-\infty}^{\infty}\left[\frac{1}{2}\left|\frac{\partial\Psi}{\partial x}\right|^2 + \frac{1}{2}|\Psi|^4 - V(x,t)|\Psi|^2\right]dx. \quad (3)$$

In the absence of driving, the Eq. (1) has a well-known solution in the shape of a moving bright soliton:

$$\Psi(x, t) = \frac{N}{2}\mathrm{sech}\left[\frac{N}{2}(x - x_0)\right]e^{iv(t)(x-x_0)}, \quad (4)$$

where $x_0(t)$ is the position of the soliton's center of mass and $v(t) = dx_0/dt$ is its velocity. If we assume that the weak ratchet potential does not affect the soliton shape during its evolution, then by substituting the expression (4) into (3) we obtain the effective Hamiltonian,

$$H_{\mathrm{eff}}(t) = p^2/(2N) + NV_{\mathrm{eff}}, \quad (5)$$

that describes the matter-wave soliton as a classical particle [12] with an effective mass [13] N and momentum $p = v(t)N$, moving in the effective potential:

$$\begin{aligned} V_{\mathrm{eff}}(x_0, t) &= \frac{1}{N}\int_{-\infty}^{\infty}|\Psi(x, 0)|^2V(x, t)dx \\ &= f(t)\frac{\pi V_0}{N}\left[\frac{\cos x_0}{\sinh(\pi/N)} + 2\frac{\cos(2x_0 + \phi)}{\sinh(2\pi/N)}\right] \\ &\equiv f(t)u(x_0). \end{aligned} \quad (6)$$

The position and velocity of the soliton in the effective-particle approximation (EPA) can be obtained from the equation of motion: $dp/dt = -\partial H_{\mathrm{eff}}/\partial x_0$.

The instantaneous shape of the effective potential, $u(x_0)$, has different symmetry properties depending on the value of N. It can be seen that for strongly localized solitons with large effective masses, $N \gg 1$, $u(x_0)$ approaches the shape of the optical lattice (2) which is asymmetric for $\phi \neq 0, \pi$. In contrast, for small N the second term in $u(x_0)$ becomes exponentially smaller than the first one, the effective potential becomes practically independent of ϕ and hence always symmetric. From these symmetry considerations we expect that the average velocity acquired by a soliton in the ratchet potential will strongly depend on N and vanish as $N \to 0$.

To test the EPA predictions we calculate the cumulative velocity $\bar{v} = (1/T)\int_0^T v(t)dt$ for solitons with different effective masses, N, both by using the effective Hamiltonian (5) and by numerical integration of the model Eq. (1). As can be seen in Fig. 1(a), where we have used $T \approx 10^3 \times 2\pi/\omega$, the EPA is in good agreement with the numerical results obtained from the GP model. As Fig. 1(a) demonstrates, for a fixed initial position of the soliton, $x_0(0)$, there is a sharp transition between a regime where a soliton oscillates between neighboring wells but is not transported [Fig. 1(b)], and a regime where the soliton acquires a ballistic motion [Fig. 1(c)]. The EPA also predicts that the velocity of the ballistic motion, $\bar{v} = -(3/2\omega)du(x_0(0))/dx_0(0)$, tends to a constant value as $N \to \infty$, which is confirmed by the numerics. Although numerical simulations employ periodic boundary conditions, the tails of the soliton never overlap.

The cumulative velocity, $\bar{v}$, is also a function of the driving frequency, ω [see inset in Fig. 1(a)]. For high frequencies the velocity vanishes as $\bar{v} \propto \omega^{-1}$, as the soliton becomes insensitive to the rapidly oscillating potential. For the driving frequencies used here the soliton with a substantial effective mass, e.g., $N = 4$, radiates very little, losing less than 1.5% of atoms for typical evolution times of $T = 5 \times 10^2$ (0.35 s). Changes of the soliton shape are also small in this regime, with the amplitude varying by less than 1.5% of the initial value, which justifies our use of Eq. (4) for the soliton profile as well as the EPA. In the limit of small ω the soliton strongly interacts with the lattice and may break up. In the EPA model the trajectory of the soliton in this case is chaotic and the study of this regime is beyond the scope of this Letter.

The dependence of the soliton velocity on its effective mass is a general feature regardless of the symmetry of the periodic potential; however, the precise form of this rela-

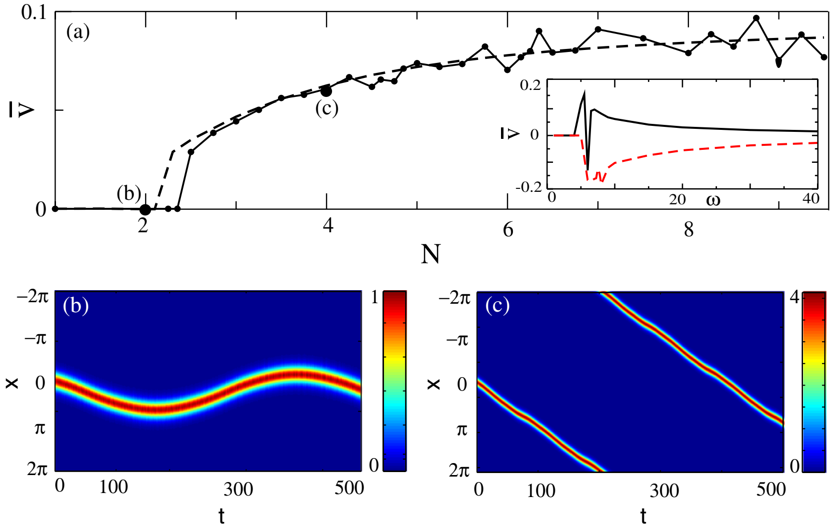

FIG. 1 (color online). (a) Cumulative velocity of a soliton, $\bar{v}$, vs effective mass, N, calculated using Eq. (1) (solid line) and EPA (dashed); $\bar{v} = 1$ corresponds to 3.5 mm/s. Parameters are: $V = 0.3$, $\phi = \pi/2$, $\omega = 10$, $x_0(0) = 0$. Inset: Cumulative velocity $\bar{v}$ vs driving frequency, ω, for $N = 4$ and $x_0(0) = 0$ (solid line) and $x_0(0) = -\pi/2$ (dashed line). (b,c) Density plot of the mean field evolution, $|\Psi(x, t)|^2$, shown for the marked points at $N = 2$ and $N = 4$ in (a).

tionship depends on the initial position of the soliton relative to the lattice. Since the space inversion symmetry of the lattice is broken, we can expect that averaging over all initial soliton positions, $x_0(0)$, will lead to a nonzero average velocity, $\langle \bar{v} \rangle = (1/2\pi) \int_0^{2\pi} \bar{v} dx_0(0)$, i.e., directed soliton current. To demonstrate this effect in Fig. 2 we plot the soliton velocity averaged over an ensemble of 40 initial positions, as a function of the effective mass. Although the average velocity is always nonzero, we can identify two different regimes of the ratchet dynamics depending on the value of N, as discussed below.

As seen in Fig. 2, for small values of N the EPA results and numerical solution of Eq. (1) disagree both on the onset of the ratchet effect and on its magnitude. For $N < 2.5$ the soliton's size is comparable to or larger than a period of the optical lattice. Hence it is more accurately described as a wave packet than an effective particle. Soliton shape changes significantly in this regime, with the amplitude varying by 5% for $N = 1$, although only 0.1% of atoms are lost to the background at $T = 5 \times 10^2$. The details of the soliton response to the flashing potential are best seen by examining the dependence of the cumulative velocity $\bar{v}$ on the initial position, $x_0(0)$. In Figs. 3(a) and 3(b) we show this response for $N = 1$ and $N = 2$. Interestingly the numerical results show that the soliton either has no cumulative velocity or moves in only one direction. As a result, for $N < 2.5$ a soliton attains a much larger average velocity than that predicted by the EPA (see Fig. 2). In fact the EPA incorrectly predicts that the soliton can move in both directions depending on its initial position. We also note that for $N = 1$ the cumulative velocity due to the EPA is almost symmetric around $x_0(0) = 0$ as expected due to the symmetry of the effective potential (6) in the limit of small N.

For large values of N, corresponding to a strongly localized matter-wave soliton, the ratchet dynamics is well described by the EPA. In Fig. 2 we observe a good agreement between the average velocity predicted by the

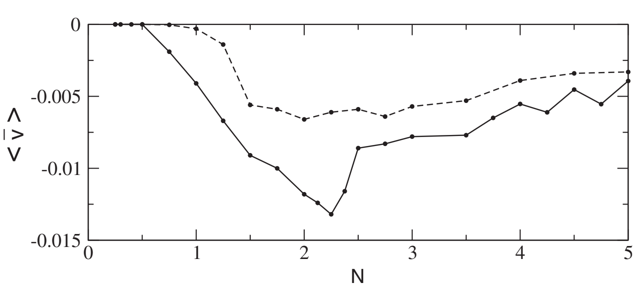

FIG. 2. (a) Average velocity of a soliton, $\langle \bar{v} \rangle$, vs effective mass N, calculated using the GP model (solid line) and EPA (dashed). Parameters are: $V_0 = 0.3$, $\phi = \pi/2$, $\omega = 10$.

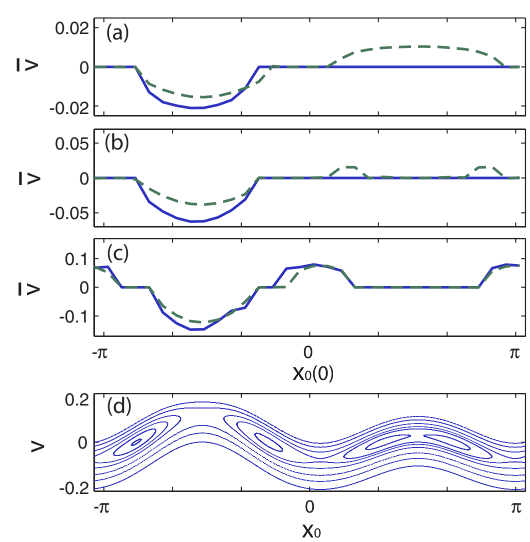

FIG. 3 (color online). (a) Cumulative velocity of a soliton, $\bar{v}$, vs initial position, $x_0(0)$, for (a) $N = 1$; (b) $N = 2$; (c) $N = 5$; calculated using the GP model (solid line) and EPA (dashed). (d) Poincaré section for the effective-particle model (5) in (c). Sections are taken at $t = 2\pi l/\omega$ where l is an integer. Parameters are: $V_0 = 0.3$, $\phi = \pi/2$, $\omega = 10$.

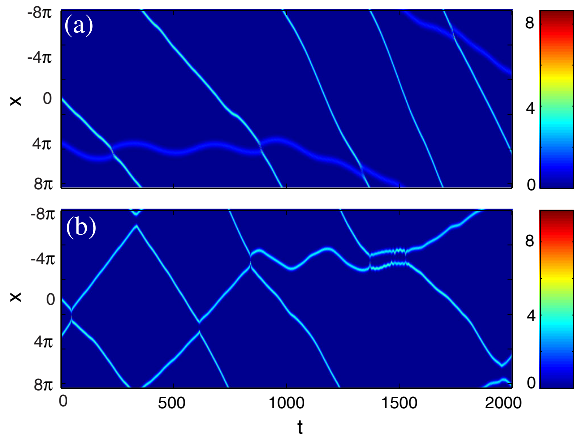

FIG. 4 (color online). Density plot, $|\Psi(x, t)|^2$, of the colliding solitons (a) with $N = 4$ and $N = 2.2$ initially located at $x_0(0) = 0$ and $x_0(0) = 4\pi$, respectively, and (b) with $N = 4$ initially located at $x_0(0) = 0$ and $x_0(0) = 3\pi + 1.2$. Parameters are: $V_0 = 0.3$. $\phi = \pi/2$, $\omega = 10$.

effective-particle model and that found from solving Eq. (1) numerically. Similarly, in Fig. 3(c) we see a good agreement between the numerics and the EPA in the details of the cumulative velocity dependence on $x_0(0)$. In Fig. 3(d) we show the Poincaré section corresponding to the EPA results of Fig. 3(c). Two different types of trajectories are observed, transporting and nontransporting. A comparison between Figs. 3(c) and 3(d) shows a clear correlation between zero (nonzero) cumulative velocity in Fig. 3(c) and a nontransporting (transporting) trajectory.

Two solitons with different N have different velocities even if their relative initial positions in the flashing lattice are the same. Therefore multiple binary collisions can be realized in a ratchet potential combined with a toroidal trap [14]. As shown in Fig. 4(a) a larger moving soliton can then induce transport of a smaller soliton which otherwise would not be transported. This happens because each collision incrementally changes the soliton's position in the phase space until it moves from a nontransporting to a transporting trajectory. The driving has little effect on the actual quasielastic scattering event.

If the solitons have equal effective masses, N, collisions can occur only if they have different relative positions in the lattice, and hence different initial velocities. In this case the interaction between the solitons is strong due to the large interaction energy [13] and each collision event induces an immediate and pronounced transition to a different phase-space trajectory. The dramatic changes in the soliton velocities after each collision are evident in Fig. 4(b). In this scenario the spatial shift that solitons acquire during each collision may lead to an effective averaging over initial positions, $x_0(0)$, after multiple scattering events. Hence, in principle, it is possible to observe a nonzero total average current for a sufficiently large number of collisions or for a sufficiently large number of interacting solitons with different initial positions.

In conclusion, we have demonstrated the ratchet dynamics of matter-wave solitons with a fixed number of atoms loaded into a time-varying potential with zero bias. The main effect of the ratchet potential is the asymmetry in the dependence of the soliton's velocity on its initial position, which results in a preferred direction of motion. The rate of transport for a given initial position is atom-number dependent, with larger solitons typically moving faster. In addition, solitons containing a small number of atoms are transported in one direction only, while larger solitons may be transported in either direction. We have established an overall directed soliton current numerically, by averaging over all initial positions. In an experiment, our main results could be confirmed either by random loading of isolated matter-wave solitons into a flashing lattice and observing a bias in cumulative velocity distribution, or by loading an array of several solitons, which would introduce effective position averaging in a single-shot experiment. Finally, we illustrated the effect of the ratchet potential on soliton scattering. Collisions can cause instantaneous transitions between nontransporting and transporting trajectories in the phase space, which could potentially be used for transport or spatial filtering of solitons based on the number of atoms.

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