Percolation Clusters above Criticality Form Kardar-Parisi-Zhang Surfaces

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This study is concerned with the characteristics of regular (isotropic) percolation clusters above the critical threshold p_c. Analytic arguments for the general dimension case, and numerical results for the two-dimensional case, lead to the conclusion that the characteristics of the shortest paths (defined as the chemical distance l) between given two sites on a percolation cluster are similar to the characteristics of optimal paths in the directed polymer model. A corollary which should be valid for the general dimension case, and verified by numerical results for the two-dimensional case, is that a cluster whose sites are at chemical distance l from a given site forms a Kardar-Parisi-Zhang surface.

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The percolation model [1–3], the directed polymer model [4], and the Kardar-Parisi-Zhang (KPZ) equation [5] are fundamental models for many real systems and the subjects of numerous studies in the past decades. The relation between the directed polymer model and the KPZ equation is well established [6], but no relation was found between the percolation model and directed polymers or KPZ surfaces. Analytic arguments and numerical results presented in this Letter indicate that there is a natural connection between the percolation model and the directed polymer model and that the percolation model should be added to Eden growth and ballistic deposition [7], as one of the main models which form KPZ surfaces. Specifically, it is shown that, above the critical threshold p_c, any shortest path on a percolation cluster has the characteristics of directed polymers and that the sites of equal shortest distance from a given site represent a KPZ surface.

Most studies of the percolation model have been focused on the characteristics of clusters near the critical threshold p_c, while the present study addresses the characteristics of clusters well above this threshold. A simple way to study the characteristics of percolation clusters is to grow them from a seed placed at $(0, 0)$ [8,9]. The present numerical study is focused on bond percolation clusters grown on a square lattice, but the conclusions are general and should be valid for the general cases of bond and site percolation and for any dimension. In the standard growth process, at each growth step l, the sites connected to the seed at step $l - 1$ are scanned, and the bonds leading from each site to its four nearest neighbors are considered: Bonds which were generated in former growth steps are left untouched, and the new neighboring bonds are generated and randomly assigned with 0 or 1, where a 0 valued bond is a connecting bond. The probability for a connecting bond is denoted by p, and in the present study p was chosen to be higher than 0.5, the critical probability (threshold) of regular (isotropic) bond percolation, but lower than 0.6447, the critical probability of directed percolation [10,11] in the diagonal direction. At each growth step l, the new sites added to the cluster are assigned with l, which is their chemical distance from the seed. (The chemical distance between two sites is defined as the length of the shortest path which connects them.) This random growth process generates a typical cluster whose sites are at a chemical distance $\leq l$ from a given site. Note that the same cluster can be generated by assigning random values only to the bonds leading from sites connected at step $l - 1$ to their unconnected nearest neighbors. In this alternative method, which was used in the present study, all of the connected sites but not all of the connecting bonds are generated. In this case, every connected site has only one and unique path which connects it to the seed.

In the directed polymer model, any site is usually connected to the origin (seed) by a large number of paths, and the optimal (minimal energy) path is defined as the path with the lowest sum of random bond values (energy). For a site at distance t from the origin, the energy standard deviation of the optimal path is proportional to t^{ω}, and the mean transversal distance of the path from the straight line which connects the site to the origin is proportional to t^{ν}. In the two-dimensional case, the values of ω and ν are $1/3$ and $2/3$, respectively. Though the original study [4] considered directed paths, which are not allowed to go backwards, it was shown [12–15] that even free optimal paths between two sites are characterized by the same growth exponents: $1/3$ and $2/3$.

It is possible to map the present case to the directed polymer case in a simple way as follows: The 1 valued bonds are assigned with a very large number, while the 0 valued bonds are assigned with 1. In this case, the shortest path of length l between two sites of the percolation cluster is also by definition the optimal path of energy l which connects them. At this stage, a distinction should be made between the situation above p_c and the situation at (or below) p_c. Above p_c the length of the shortest path which connects two sites at distance R is $\sim R$, while at (or below) p_c such a path is a fractal whose length is $\sim R^{D_f}$, $D_f > 1$.

Note that this distinction is similar to the one presented in Refs. [12,13] for the strong disorder distribution [rather than the (0, 1) distribution]. As a consequence, the shortest paths above p_c should present the characteristics of the minimal energy paths in the directed polymer model, while such paths at (or below) p_c are completely different. The numerical results presented below for $p > p_c$ support the above arguments.

Most of the numerical results were obtained for $p = 0.5529$. Obviously, these results should be valid for all of the range of p values between 0.5 and 0.6447. (The cases of $p \leq 0.5$ and $p > 0.6447$ are discussed at the end.)

The variables whose values were numerically estimated are as follows.

(1) $l_d(t)$: The number of growth steps at the first arrival to a site with coordinates (i, j), where $|i| + |j| = t$. These sites form the four sides of the square of diagonals between $(\pm t, 0)$ and $(0, \pm t)$. [The value of $p = 0.5529$ was chosen because at this probability $\overline{l_d(t)} \simeq 1.1t$.]

(2) $l_v(t)$: The number of growth steps at the first arrival to a site with coordinates (i, j), where $\mathrm{Max}(|i|, |j|) = t$. These sites form the four sides of the square lattice whose edges are at $(-t, t)$, (t, t), $(t, -t)$, and $(-t, -t)$.

(3) $l_{\min}(t)$: The number of growth steps at the first arrival to a site with coordinates (i, j), where $\sqrt{(i^2 + j^2)} \geq t$. The relevant sites (nearly) form a circle of radius t around the seed.

(4) $D_d(t)$: The distance between the site (i, j) with $|i| + |j| = t$, mentioned at (1), and the coordinate $(\pm t/2, \pm t/2)$, which is at the center of the relevant diagonal side.

(5) $D_v(t)$: The distance between the site (i, j), where $\mathrm{Max}(|i|, |j|) = t$, mentioned at (2), and the coordinate $(\pm t, 0)$ or $(0, \pm t)$, which is at the center of the relevant vertical or horizontal side.

(6) $R_d(l)$: The maximal diagonal distance of the cluster from its seed, after l steps of growth. This distance is defined by the maximal value of $\sqrt{2}t$ of the sites of the cluster whose coordinates are $(\pm t, \pm t)$.

(7) $R_v(l)$: The maximal horizontal or vertical distance of the cluster from its seed, after l steps of growth. This distance is defined by the maximal value of t of the sites of the cluster whose coordinates are $(\pm t, 0)$ or $(0, \pm t)$.

(8) $R_{\max}(l)$: The maximal geometrical distance $\sqrt{i^2 + j^2}$ from the seed to any of the sites of the cluster after l steps of growth.

Note that the number of growth steps at the first arrival to a site is identical to the length of the shortest path between this site and the origin. In view of the mapping presented above, the characteristics of these shortest paths should be similar to the characteristics of the minimal energy paths in the directed polymer model.

Consider first $l_d(t)$, $l_v(t)$, and $l_{\min}(t)$ and denote by $\sigma(t)$ their standard deviations. The numerical results indicate that these standard deviations are proportional to t^{ω}, and the local values of the exponent ω, computed by $\log_2[\sigma(t)/\sigma(t/2)]$ are presented in Fig. 1.

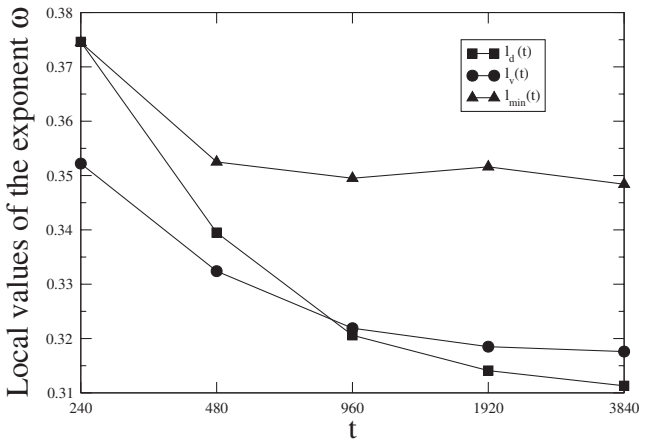

FIG. 1. The local values of the exponent ω estimated for $l_d(t)$ (squares), $l_v(t)$ (circles), and $l_{\min}(t)$ (triangles).

Obviously, for large enough t, the three curves should converge to a common asymptotic value, and their difference is an outcome of finite size effects. The data presented in the figure point at an asymptotic value which is in the vicinity of 1/3, the directed polymer value. It should be emphasized that the error bars associated with the results presented in this figure, and in all of the following figures, are smaller than the size of the symbols, and thus the results are practically error-free. Even the slight rise and fall of the upper curve of Fig. 1 is an outcome of finite size effects and not of estimation errors.

The similarity between the shortest paths in the present case and the lowest energy paths in the directed polymer case implies that $D_d(t)$ and $D_v(t)$ should be proportional to t^{ν}, where ν has the directed polymer value of 2/3. The local values of the exponent ν estimated for these variables are presented in Fig. 2.

In this case the values are very close to 2/3, and the influence of finite size effects is weak compared to the data presented in Fig. 1. The fact that, in the directed polymer model, the local values of ν are much less sensitive to finite

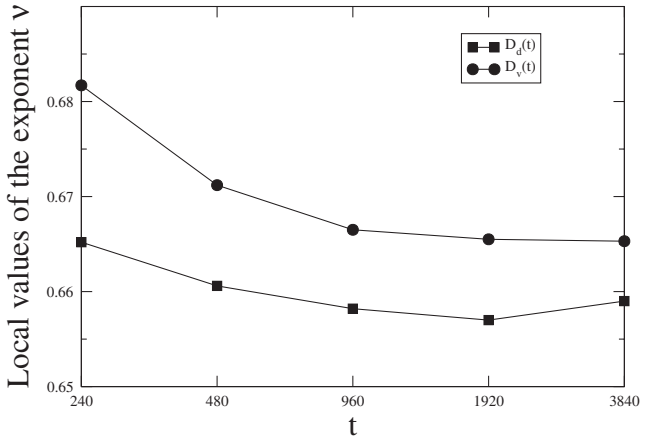

FIG. 2. The local values of the exponent ν estimated for $D_d(t)$ (squares) and $D_v(t)$ (circles).

size effects than the local values of ω was already noticed in Ref. [16].

The above discussion refers to the characteristics of the shortest paths between the seed at the center and predetermined lines (or a curve) on the lattice. The shape of the cluster whose sites are at chemical distance $\leq l$ from the seed is determined by the variable $R_\theta(l)$, which is the maximal distance of the sites of this cluster in the direction of the angle θ. In the numerical study, it was found that $\overline{R_\theta(l)}$ has a maximum at $\theta = 0, 90, 180, 270$ and $\mathrm{Max}[R_0(l), R_{90}(l), R_{180}(l), R_{270}(l)]$ is denoted above by $R_v(l)$. The minimum of $\overline{R_\theta(l)}$ is attained at $\theta = 45, 135, 225, 315$, and $\mathrm{Max}[R_{45}(l), R_{135}(l), R_{225}(l), R_{315}(l)]$ is denoted above by $R_d(l)$. The numerical results indicate that at $p = 0.5529$ the ratio $\overline{R_v(l)}/\overline{R_d(l)} \simeq 1.013$, suggesting that the shape of the cluster is close to, but not exactly, a perfect circle. Naturally, the value of this ratio depends on p: The closer p is to 0.5, the closer this ratio is to 1.

The roughness of the circumference of the cluster is determined by the standard deviation of $R_\theta(l)$ and by the lateral correlations along the circumference. Note that the minimal number of growth steps needed to arrive at a fixed distance from the seed and the maximal distance from the seed arrived after a fixed number of growth steps are complementary variables, and it can be expected that their standard deviations would have the same growth exponents and thus that the surface of a percolation cluster whose sites are at chemical distance $\leq l$ from a given site forms a KPZ surface. (Compare to the results related to the Eden growth model presented in [12].)

In order to find the dependence of the standard deviation of $R_\theta(l)$ on l, the standard deviations of $R_d(l)$, $R_v(l)$, and $R_{\max}(l)$ were estimated, and, as expected, the numerical results indicate that their dependence on l is similar to the one presented in Fig. 1. The local values of the exponent ω estimated for these three variables are presented in Fig. 3, and the only significant difference between Figs. 1 and 3 is the final rise of the two lower curves in Fig. 3, which supports the conclusion that the common asymptotic value of all six curves presented in Figs. 1 and 3 is $1/3$.

The lateral correlations of KPZ surfaces are known to be proportional to $R^{2/3}$ [6]. In view of the results presented in Fig. 2, it can be expected that the lateral correlations in the present percolation cluster case should also grow in a similar way. The lateral correlations were defined by $[\overline{R(0)R(i)} - \overline{R(0)}\,\overline{R(i)}]/\{\sigma[R(0)]\sigma[R(i)]\}$, where $R(i)$ is the maximal geometrical distance between the seed and the sites whose horizontal coordinate is i and $\sigma(R)$ is the standard deviation of the distance R. These lateral correlations were estimated for $i = 1, 2, 4, 8, \ldots$, etc. For $i \ll R$, there is no significant difference between i and the distance along the circumference of the circle whose radius is R. Note that, if the correlations grow in proportion to $l^{2/3}$, the values recorded for (l, i) and $(l \times 2^{3/2}, 2i)$ should be equal. The correlations estimated for $l = 2168$ and $l = 6132 \simeq 2168 \times 2^{3/2}$ are presented in Fig. 4.

As can be seen, for $1 \leq i \leq 64$ the correlations estimated for $(2168, i)$ and $(6132, 2i)$ are quite close, and the differences are attributed to finite size effects (and not to estimation errors). Of course, the results obtained for $i \geq 128$ are mainly affected by the relatively small value of l and not by finite size effects.

In the case that $p \leq p_c$, the situation is completely different. After l growth steps, the cluster is composed of few and scattered strands of sites, and the surface of the cluster has an erratic shape which certainly does not present KPZ characteristics.

In the case that p is higher than 0.6447, the diagonal directed percolation p_c, there is a difference between the diagonal directions and the horizontal or vertical directions. In the diagonal direction, there are many possible paths of length t which lead to the site $(t/2, t/2)$, and thus the mean minimal number of growth steps needed to arrive at sites whose $|i| + |j| = t$ is $t + k$, where k is a constant of the order of 1. As a consequence, the standard deviation of the length of the shortest paths is a constant independent of

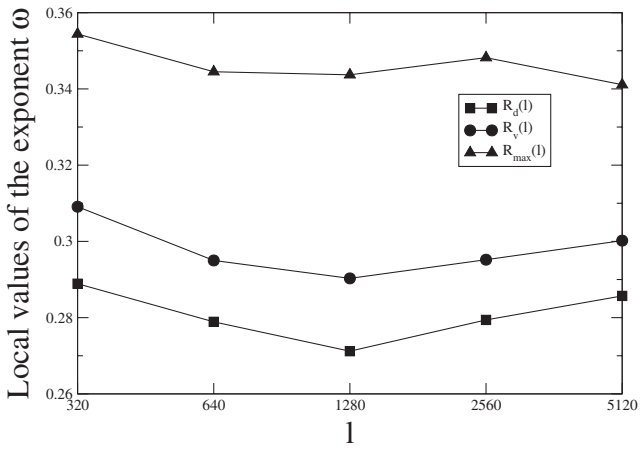

FIG. 3. The local values of the exponent ω estimated for $R_d(l)$ (squares), $R_v(l)$ (circles), and $R_{\max}(l)$ (triangles).

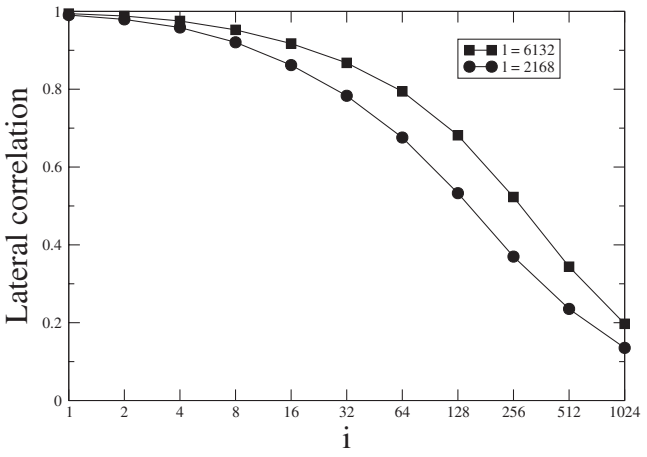

FIG. 4. The lateral correlations estimated for $l = 6132$ (squares) and $l = 2168$ (circles).

t, and the surface near the sites (i, j) where $|i| = |j|$ is nearly flat and does not present KPZ characteristics. In the horizontal or vertical directions there is only one path of length t which leads to the sites $(\pm t, 0)$ or $(0, \pm t)$, and thus the mean minimal number of growth steps needed to arrive at the sites $(t, 0)$ or $(0, t)$ is $t(1 + k) = t + kt$, where k is a constant whose value depends on p. In this case there is no constant upper bound on the standard deviation of the length of the shortest paths, it can and does grow in proportion to $t^{1/3}$, and the surface near the sites $(t, 0)$ and $(0, t)$ is a KPZ surface. The same conclusion holds for directed percolation clusters whose p is higher than their relevant p_c. In the diagonal direction they are nearly flat (in the middle), and in the horizontal or vertical direction they form KPZ surfaces.

As a final remark, it should not be forgotten that in the present case the clusters are considered at $p > p_c$. Thus, the fact that the percolation model has an upper critical dimension does not imply an upper critical dimension for KPZ surfaces.

In conclusion, the numerical results and analytic arguments presented above indicate that, in percolation clusters whose $p > p_c$, the sites which are at a chemical distance $\leq l$ from a given site form nearly perfect circles whose circumference roughness is of the KPZ type.

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