Supersolitons: Solitonic Excitations in Atomic Soliton Chains

David Novoa,[1] Boris A. Malomed,[2] Humberto Michinel,[1] and Víctor M. Pérez-García[3]

[1]*Área de Óptica, Facultade de Ciencias, Universidade de Vigo, As Lagoas s/n, Ourense, E-32004 Spain*
[2]*Department of Physical Electronics, School of Electrical Engineering, Faculty of Engineering, Tel Aviv University, Tel Aviv 69978, Israel*
[3]*Departamento de Matemáticas, E.T.S.I. Industriales, and Instituto de Matemática Aplicada a la Ciencia y la Ingeniería, Universidad de Castilla-La Mancha, Avenida Camilo José Cela, 3, 13071 Ciudad Real, Spain*

(Received 11 April 2008; published 29 September 2008)

We show that, by tuning interactions in nonintegrable vector nonlinear Schrödinger equations modeling Bose-Einstein condensates and other relevant physical systems, it is possible to achieve a regime of elastic particlelike collisions between solitons. This would allow one to construct a Newton's cradle with solitons and *supersolitons*: localized collective excitations in solitary-wave chains.

DOI: 10.1103/PhysRevLett.101.144101 PACS numbers: 05.45.Yv, 03.75.Lm

Introduction and model.—One of the most successful concepts of nonlinear science, with applications to a great variety of physical contexts, is that of solitons, i.e., localized waves sustained by the balance between dispersion and nonlinearity. Many types of solitons have been studied, starting with classical examples found in integrable models, such as the Korteweg–de Vries (KdV), sine-Gordon, Toda-lattice (TL), and nonlinear Schrödinger (NLS) equations, and their nonintegrable extensions.

Solitons are robust against collisions due to the integrability of the underlying equations. A lot of activity has been directed at the study of soliton collisions and interactions in nonintegrable systems. Recent advances include the analysis of chaotic scattering [1], the formation of bound states and clusters of solitons [2,3], and soliton collisions in vectorial systems [1,4,5].

While it is customary to speak of solitons as elastically colliding quasiparticles, they clearly feature the underlying wave structure while passing through each other, especially in integrable systems. However, elastic collisions between classical rigid particles lead to a momentum exchange between them and rebound [6]. In this work, we propose a soliton-collision scenario of physical relevance, which allows a realization of truly elastic particlelike collisions far from integrability. We will show how this can be used to build a vectorial-soliton version of the set of adjacent classical pendula known as *Newton's cradle* and to create *supersolitons*, i.e., collective solitonlike excitations over arrays of solitary waves, leading to a remarkable conjunction of two generic phenomena: the formation of robust soliton trains [7] and the emergence of quasidiscrete solitons at a higher level of organization.

The paradigmatic model which allows us to implement the above-mentioned effects is based on the two-component NLS equation, arising in sundry contexts [8],

$$i\frac{\partial u_j}{\partial t} = -\frac{1}{2}\frac{\partial u_j}{\partial x^2} + V(x)u_j + \sum_{k=1,2} g_{jk}|u_k|^2 u_j. \quad (1)$$

A direct realization of this model is a two-component, alias binary ($j = 1, 2$), Bose-Einstein condensate (BEC), where u_j are wave functions of two atomic states with mass m, under a strong transverse confinement of frequency $\nu_\perp$ [9]. The spatial variable x and time t are measured, respectively, in units of $a_0 \equiv \sqrt{\hbar/m\nu_\perp}$ and $1/\nu_\perp$, while $g_{jk} \equiv 2a_{jk}/a_0$ are given in terms of the s-wave scattering lengths a_{jk}, and the numbers of atoms in the two species are proportional to $\int_{-\infty}^{+\infty}|u_j|^2 dx$. Matter-wave solitons in BECs have been created experimentally [10] and their interactions studied theoretically in detail [11–13].

Particlelike elastic collisions and the solitonic Newton's cradle.—We consider Eqs. (1) with intracomponent attraction ($g_{11}, g_{22} < 0$) and intercomponent repulsion ($g_{12}, g_{21} > 0$). We choose $g_{11} = g_{22} = -g_{12} = -g_{21} \equiv 1$ (without loss of generality), for which case the NLS system is far from the integrability [14]. In this situation, the solitons belonging to different components interact incoherently with a repulsive force, and thus collisions between such solitons will resemble those between elastic beads. We will consider models with both harmonic longitudinal confinement $V(x) = \Omega^2 x^2/2$ and ring-shaped configurations [15]. Trains of N solitons ($n = 1, \ldots, N$) of two atomic species ($j = 1, 2$) will be taken as superpositions of (initially) far separated pulses which, in isolation, are single-soliton solutions to either equation (1), with $j = 1$ or 2, in the absence of the longitudinal potential:

$$u_j(x, 0) = \sum_{n=1,\ldots,N} (-1)^n \mathrm{sech}(x - \xi_j^{(n)}) e^{ixv_j^{(n)}}. \quad (2)$$

We consider chains of alternating solitons in the two species, with separation Δ, i.e., $\xi_1^{(n)} = n\Delta$, $\xi_2^{(n)} = (n + \frac{1}{2})\Delta$ and initial velocities $v_j^{(n)}$. In Fig. 1, where the potential is absent, panel (a) displays a single-collision event ($N = 1$). Because of the repulsive character of the intercomponent interaction, the incident soliton (in field

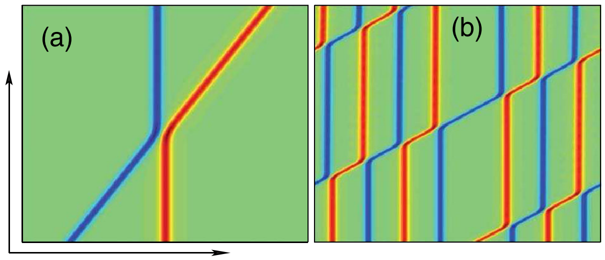

FIG. 1 (color online). Pseudocolor plots of $|u_1(x,t)|^2$ (blue) and $|u_2(x,t)|^2$ (red) display the evolution of initially well separated soliton trains built as per Eq. (2). (a) Collision of two solitons ($N=1$) for $\xi_1^{(1)}=-20$, $\xi_2^{(1)}=0$, $v_1^{(1)}=0.4$, $v_2^{(1)}=0$ $-30<x<+30$, and $0<t<100$. (b) Multisoliton collisions in a ring with $N=4$ and $\Delta=20$, for $\xi_2^{(1)}=-35$, $\xi_1^{(1)}=-25$, and zero input velocities, except for $v_2^{(3)}=0.5$. The displayed domain is $-40<x<+40$ and $0<t<250$.

u_1) transfers all of its momentum to the initially quiescent soliton (in u_2), in full analogy to the behavior of elastic particles and contrary to the typical behavior of nontopological solitons in integrable systems. The dynamics of a ring chain of eight alternating solitons shows the periodic transfer of momentum in Fig. 1(b).

The parabolic trapping potential compels the solitons to oscillate around equilibrium positions. In the quantum counterpart of the model, this setup opens a way to build a *quantum Newton's cradle* made of matter-wave solitons; see Fig. 2. Unlike other settings explored in BEC [16], the cradle configuration does not require a lattice potential to create effective particles, which are here created solely by the nonlinear interactions.

The Toda-lattice limit: Supersolitons.—We get back to the setting based on two alternating chains of solitons set along a ring. Within the adiabatic approximation, which assumes no deformation of the solitons and identical amplitudes in each component, they are approximated by

$$u_j^{(n)}(x,t) = \eta_j \mathrm{sech}[\eta_j(x-\xi_j^{(n)})] \tag{3}$$

$$\times \exp\left\{ i\dot{\xi}_j^{(n)}x + \frac{i}{2}\int[\eta_j^2 - (\dot{\xi}_j^{(n)})^2]dt \right\}, \tag{4}$$

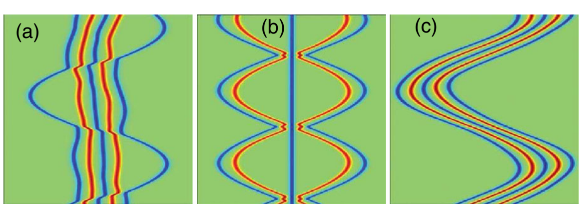

FIG. 2 (color online). Same as Fig. 1 for the *Newton's cradle* built of five solitons, in the presence of the external potential with $\Omega^2=6\times10^{-5}$. In all subplots, the initial separation between the solitons is $\Delta=12$, and the domain is $0<t<1000$, $-40<x<+40$. Different oscillation modes are excited by imparting the initial velocity, of size 0.2, to (a) the top-left soliton, (b) two ultimate solitons on both sides, with opposite velocities, and (c) all solitons.

η_j and $\dot{\xi}_j^{(n)}$ being the peak amplitude and velocity of the nth soliton in the jth component, respectively. An analysis based on the soliton perturbation theory [17] yields the equations of motion for the soliton coordinates:

$$\ddot{\xi}_j^{(n)} = -\Omega^2\xi_j^{(n)} + 8\eta_{3-j}\sum_{k=1,2}\eta_k^2[\exp(\phi_k^{(j)}) - \exp(\theta_k^{(j)})], \tag{5}$$

where $j=1,2$, $\phi_k^{(j)} \equiv (-1)^j 2\eta_k(\xi_1^{(n)} - \xi_2^{(n+j-2)})$, and $\theta_k^{(j)} \equiv (-1)^j 2\eta_k(\xi_2^{(n)} - \xi_1^{(n+j-1)})$. These equations take into regard only the interactions between nearest neighbors in the binary chain, as the large separation between the solitons makes the second-nearest-neighbor interactions (within each component) negligible. Thus, the evolution equations for solitons' phases keep their unperturbed (single-soliton) form, and Eqs. (5) do not include equations for the amplitudes as in other approaches (see, e.g., [3]), since they remain constant.

Equations (5) are expected to be accurate when the separation between adjacent solitons essentially exceeds their widths. Similar ideas have been used to derive equations for the interaction of solitons of other types [4,7,13,18]. When $\Omega=0$, the model reduces to the so-called diatomic TL, which is *not* integrable, although some solutions are known for it [19].

Setting $\eta_1=\eta_2\equiv\eta$ and $\Omega=0$ in Eqs. (5), and defining displacements of the solitons from their equilibrium positions, $q_{2n}(t)\equiv 2\eta[\xi_1^{(n)}(t) - nL/N]$ and $q_{2n+1}(t)\equiv 2\eta[\xi_2^{(n)}(t) - (2n+1)L/(2N)]$, where L is the total length of the ring, we get the integrable TL [20]

$$\ddot{q}_n = \alpha\exp[-(q_n - q_{n-1})] - \alpha\exp[-(q_{n+1}-q_n)], \tag{6}$$

where $\alpha\equiv 32\eta^4 e^{-\eta L/N}$. It gives rise to a family of exact solutions $q_n(t) = q(n-ct)$ for solitons running across the lattice with the normalized velocity that takes values $|c| > c_{\min}\equiv\sqrt{\alpha}$. In the limit of $|c|\gg\sqrt{\alpha}$, the TL soliton reduces to a single bead moving in a chain of separated rigid beads. However, in real condensed-matter media, potentials of the interaction between adjacent atoms are not exponential, unlike Eq. (6), being closer to those corresponding to anharmonic oscillators. This is why the only experimental realization of the integrable TL was reported in electric transmission lines [21] that may be designed in exact correspondence to Eqs. (6). Our analysis suggests a possibility to create Toda solitons, of both mono- and diatomic types, as excitations in interwoven arrays of matter-wave solitons in binary BECs.

We name these excitations *supersolitons*, as they are predicted to occur on top of the array of "elementary" solitons and are expected to be as robust as solitons in integrable models. In a completely different context, the same name was previously applied to solitons in supersymmetric models [22] and, which is closer to the present context but is nevertheless quite different, to *topological* collective excitations in chains of fluxons trapped in periodically inhomogeneous Josephson junctions or layered superconducting structures [18]. Actually, the TL supersolitons represent a higher tier of the soliton hierarchy, built on top of solitary waves of the *strongly nonintegrable* binary NLS system (unlike soliton trains in the integrable NLS equation [7]).

For small perturbations with frequency ω and wave number p, Eq. (6) gives rise to a dispersion relation which determines the phase velocity of linear waves: $c_{\rm ph} \equiv \omega/p = (2\sqrt{\alpha}/p)\sin(p/2)$. The wave number quantization imposed by the ring geometry, $p = \pi M/N$, $M = 0, \pm1, \pm2, \ldots$, leads to the discrete velocity spectrum

$$|c_{\rm ph}^{(M)}| = (2\sqrt{\alpha}N/\pi M)\sin(\pi M/2N) < c_{\rm min} = \sqrt{\alpha}. \quad (7)$$

The excitation of the array by kicking a soliton with velocity v_0 may generate a supersoliton, if

$$|c_0| \equiv (2N/L)|v_0| > c_{\rm min}. \quad (8)$$

In the opposite case, $|c_0| < c_{\rm min}$, the kick excites small-amplitude waves. We expect that the latter effect will be amplified if c_0 is close to any *resonant value* (7); cf. Ref. [23]. In this Letter, we focus on the study of the creation and dynamics of supersolitons. Resonant effects in the excitation of linear waves will be reported elsewhere.

Numerical studies of supersolitons.—In Fig. 3, we demonstrate the generation of a single supersoliton by kicking one soliton in either component. Since the excitation does not exactly correspond to a supersoliton, a small amount of radiation is generated in the form of "ripples" propagating in the soliton lattice. The transmission of the supersoliton is nearly perfect, as seen from the amplitude plot in Fig. 3(a) and the full propagation picture in Fig. 3(b). With the initial amplitude of the solitons in the chain $\eta = 1$ and separation between them $L/2N = 5$, Eqs. (6) and (8) yield the threshold (minimum) velocity for the TL supersoliton $(L/2N)c_{\rm min} \approx 0.19$, which is below the kick used here, $|v_0| = 0.5$. A minor effect observed in Fig. 3(a) and not considered in our model is weak compression of individual solitary waves when they are hit by the supersoliton.

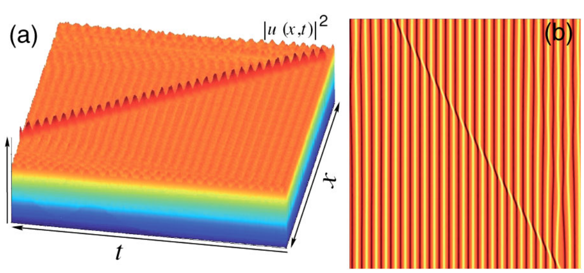

FIG. 3 (color online). Generation of a supersoliton in the chain of individual solitons of equal amplitudes $\eta = 1$ in each component, built as per Eq. (2), with $\Delta = 10$ and $N = 24$, by kicking a single soliton with velocity $v_2^{(22)} = -0.5$. The ring's length is $L = 240$. (a) Spatiotemporal plot of $|u(x,t)|^2 \equiv |u_1(x,t)|^2 + |u_2(x,t)|^2$. (b) Pseudocolor plots showing $|u_1(x,t)|^2$ (yellow) and $|u_2(x,t)|^2$ (red). The time interval is $0 < t < 250$.

The dynamics of the supersolitons is further illustrated by Fig. 4, which shows elastic head-on and overtaking collisions between two supersolitons. We stress again that these behaviors, characteristic of integrable systems, arise in chains of nonintegrable solitary waves.

Can scalar models support supersolitons?—Soliton collisions in the framework of *scalar* NLS equations have been studied in various contexts, and the so-called complex TL equation has been derived using different approaches [4,13]. However, trains of equal-amplitude solitons in the framework of that model turn out to be *unstable* because of the phase dependence of the interactions [24]. An example displayed in Fig. 5(a) shows that the initial kick generates unstable dynamics in the single-component chain, whereas its alternating binary counterpart does not display any instability in Fig. 5(b).

Experimental realization.—The possibility to create supersolitons in binary BECs depends on the use of the Feshbach-resonance techniques to properly tune attractive intraspecies and repulsive interspecies interactions in the mixture. Atomic mixtures with precisely controllable interspecies interactions have been recently reported in Ref. [25]. The initial state with alternating solitons, necessary for the implementation of our scheme, may be created using modulational instability and segregation in an initially stable binary condensate [26].

Conclusions.—We have explored a physical model based on the two-component NLS equation which gives rise to elastic particlelike collisions between solitons belonging to different species. These interactions make it possible to create an analog of the Newton's cradle and supersolitons, leading to the prediction of effectively integrable dynamics on top of arrays of solitary waves in strongly nonintegrable subsystems.

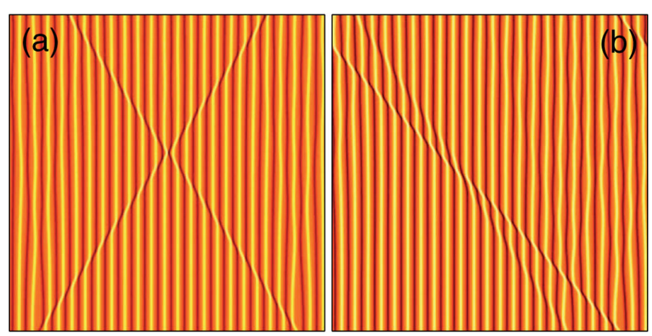

FIG. 4 (color online). Collisions between supersolitons created from the same initial configurations as in Fig. 3. (a) A head-on collision induced by kick parameters $v_1^{(3)} = -v_2^{(22)} = 0.5$. (b) An overtaking collision, with initial velocities $v_1^{(23)} = -0.6$ and $v_2^{(19)} = -0.3$. Time intervals in the two panels are $0 < t < 250$ and $0 < t < 350$, respectively.

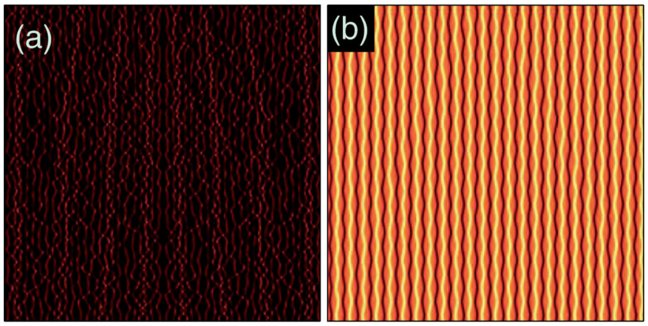

FIG. 5 (color online). The evolution, for $0 < t < 500$, of soliton trains in the case of (a) one and (b) two components, which are described, respectively, by the scalar and coupled NLS equations (black and white lines). The excitation is introduced by kicking adjacent solitons with opposite velocities: $v = \pm 0.1$ in (a) and $v_1^{(n)} = 0.1$ and $v_2^{(n)} = -0.1$ in (b). Other parameters are as in Fig. 3.

Our results suggest intriguing questions to explore in the future such as the possibility of constructing supersolitons using dark solitons as building blocks, the extension of our results to multidimensional scenarios, or the possibility of using solitons of different amplitudes to enhance the stability of the soliton trains as was shown in Ref. [27] for the scalar case.

This work has been partially supported by Grants No. FIS2006-04190, No. FIS2004-02466, No. FIS2007-29090-E (Ministerio de Educación y Ciencia, Spain), No. PGIDIT04TIC383001PR (Xunta de Galicia), and No. PCI-08-0093 (Junta de Comunidades de Castilla-La Mancha, Spain). D. N. acknowledges a grant from Consellería de Educación (Xunta de Galicia).

[1] J. K. Yang and Y. Tan, Phys. Rev. Lett. **85**, 3624 (2000); S. V. Dmitriev *et al.*, Phys. Rev. E **68**, 056603 (2003); R. H. Goodman and R. Haberman, Phys. Rev. Lett. **98**, 104103 (2007).

[2] J. K. Yang *et al.*, Phys. Rev. Lett. **94**, 113902 (2005); M. Stratmann, T. Pagel, and F. Mitschke, *ibid.* **95**, 143902 (2005); Y. Leitner and B. A. Malomed, Phys. Rev. E **71**, 057601 (2005); C. Rotschild *et al.*, Nature Phys. **2**, 769 (2006); D. Buccoliero *et al.*, Phys. Rev. Lett. **98**, 053901 (2007); J. Belmonte-Beitia *et al.*, *ibid.* **98**, 064102 (2007).

[3] V. S. Gerdjikov, E. N. Doktorov, and N. P. Matsuka, Theor. Math. Phys. **151**, 762 (2007).

[4] V. S. Gerdjikov and I. M. Uzunov, Physica (Amsterdam) **152D**, 355 (2001).

[5] T. Kanna *et al.*, Phys. Rev. E **73**, 026604 (2006); T.-S. Ku *et al.*, Phys. Rev. Lett. **94**, 063904 (2005).

[6] In the KdV equation, a specific decomposition of two-soliton solutions makes it possible to interpret solitons as quasirigid objects: A. Benes, N. Kasman, and K. Young, J. Nonlinear Sci. **16**, 179 (2006), and references therein.

[7] V. S. Gerdjikov *et al.*, Phys. Lett. A **241**, 323 (1998).

[8] Y. Kivshar and G. P. Agrawal, *Optical Solitons: From Fibers to Photonic Crystals* (Academic Press, New York, 2003); L. P. Pitaevskii and S. Stringari, *Bose-Einstein Condensation* (Oxford University Press, Oxford, 2003).

[9] V. M. Pérez-García, H. Michinel, and H. Herrero, Phys. Rev. A **57**, 3837 (1998).

[10] S. Burger *et al.*, Phys. Rev. Lett. **83**, 5198 (1999); B. Eiermann *et al.*, *ibid.* **92**, 230401 (2004); G. B. Partridge, A. G. Truscott, and R. G. Hulet, Nature (London) **417**, 150 (2002); L. Khaykovich *et al.*, Science **296**, 1290 (2002); C. Becker *et al.*, Nature Phys. **4**, 496 (2008).

[11] L. D. Carr and J. Brand, Phys. Rev. Lett. **92**, 040401 (2004); V. M. Pérez-García and J. B. Beitia, Phys. Rev. A **72**, 033620 (2005); J. Babarro *et al.*, *ibid.* **71**, 043608 (2005); L. Khaykovich and B. A. Malomed, *ibid.* **74**, 023607 (2006).

[12] V. S. Gerdjikov, B. B. Baizakov, and M. Salerno, Theor. Math. Phys. **144**, 1138 (2005).

[13] B. B. Baizakov, B. A. Malomed, and M. Salerno, Phys. Rev. A **70**, 053613 (2004); A. D. Martin, C. S. Adams, and S. A. Gardiner, Phys. Rev. Lett. **98**, 020402 (2007).

[14] V. E. Zakharov and E. I. Schulman, Physica (Amsterdam) **4D**, 270 (1982).

[15] S. Gupta *et al.*, Phys. Rev. Lett. **95**, 143201 (2005); C. Ryu *et al.*, Phys. Rev. Lett. **99**, 260401 (2007).

[16] T. Kinoshita, T. Wenger, and D. S. Weiss, Nature (London) **440**, 900 (2006).

[17] Yu. S. Kivshar and B. A. Malomed, Rev. Mod. Phys. **61**, 763 (1989).

[18] A. V. Ustinov, Phys. Lett. A **136**, 155 (1989); B. A. Malomed, Phys. Rev. B **41**, 2616 (1990); Y. S. Kivshar and T. K. Soboleva, Phys. Rev. B **42**, 2655(R) (1990).

[19] S. Lepria, R. Livi, and A. Politi, Phys. Rep. **377**, 1 (2003).

[20] M. Toda, *Theory of Nonlinear Lattices* (Springer-Verlag, New York, 1989).

[21] See P. Marquie, J. M. Bilbault, and M. Remoissenet, Phys. Rev. E **51**, 6127 (1995), and references therein.

[22] Yu. I. Manin and A. Radul, Commun. Math. Phys. **98**, 65 (1985); A. S. Carstea, Nonlinearity **13**, 1645 (2000); Y. Ya-Xuan, Commun. Theor. Phys. **49**, 685 (2008).

[23] A. V. Ustinov, M. Cirillo, and B. A. Malomed, Phys. Rev. B **47**, 8357 (1993).

[24] T. Gallay and M. Haragus, J. Dyn. Differ. Equ. **19**, 825 (2007), and references therein.

[25] G. Thalhammer *et al.*, Phys. Rev. Lett. **100**, 210402 (2008).

[26] K. Kasamatsu and M. Tsubota, Phys. Rev. Lett. **93**, 100402 (2004); Phys. Rev. A **74**, 013617 (2006).

[27] V. S. Gerdjikov *et al.*, Phys. Lett. A **241**, 323 (1998).