Broadband Cylindrical Acoustic Cloak for Linear Surface Waves in a Fluid

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(Received 30 March 2008; published 25 September 2008)

We describe the first practical realization of a cylindrical cloak for linear surface liquid waves. This structured metamaterial bends surface waves radiated by a closely located acoustic source over a finite interval of Hertz frequencies. We demonstrate theoretically its unique mechanism using homogenization theory: the cloak behaves as an effective anisotropic fluid characterized by a diagonal stress tensor in a cylindrical basis. A low azimuthal viscosity is achieved, where the fluid flows most rapidly. Numerical simulations demonstrate that the homogenized cloak behaves like the actual structured cloak. We experimentally analyze the decreased backscattering of a fluid with low viscosity and finite density (methoxynonafluorobutane) from a cylindrical rigid obstacle surrounded by the cloak when it is located a couple of wavelengths away from the acoustic source.

DOI: 10.1103/PhysRevLett.101.134501 PACS numbers: 47.35.Lf, 47.11.Fg, 47.11.St

The transformation based solutions to the Maxwell equations in curvilinear coordinate systems reported by Pendry *et al.* in [1] enables one to bend electromagnetic waves around arbitrarily sized and shaped solids. The electromagnetic invisibility cloak is a metamaterial which maps a concealment region into a surrounding shell: as a result of the coordinate transformation the permittivity and permeability are strongly heterogeneous and anisotropic within the cloak, yet fulfilling impedance matching with the surrounding vacuum. The cloak thus neither scatters waves nor induces a shadow in the transmitted field. In [2], a cylindrical electromagnetic cloak was constructed using specially designed concentric arrays of split ring resonators which enable to meet among others the prerequisite artificial magnetism property [3]. This locally resonant microstructured cloak was shown to conceal a copper cylinder around 8.5 GHz, as predicted by numerical simulations [2]. Interestingly, there are two different solutions to Maxwell's equations for an object with and without the cloak that have in theory the exact same field distributions outside the cloak. But as noted in [2], this is not in contradiction with the uniqueness theorem which only applies to isotropic media [4]. The effectiveness of the transformation based cloak was demonstrated theoretically and numerically solving the Schrödinger equation which is valid in the geometric optic limit [5], and solving Maxwell's equations using finite elements for an incident plane wave (far field limit) [6] and a line current source (near field limit) [7]. In [8], a reduced set of material parameters was introduced to relax the constraint on the permeability, necessarily leading to an impedance mismatch with vacuum which was shown to preserve the cloak effectiveness to a good extent. Other routes to invisibility include reduction of backscatter [9] and cloaking through anomalous localized resonances, the latter one using negative refraction [10,11]. It was further shown that the transformation based invisibility cloaks could be applied to certain types of elastodynamic waves in structural mechanics [12]. Such neutral inclusions have been also studied in the elastostatic context using asymptotic and computational methods in the case of antiplane shear and in-plane coupled pressure and shear polarizations [13]. Cummer and Schurig demonstrated that acoustic waves in a fluid also undergo the same geometric transform for a 2D geometry [14], which has been since then generalized to 3D acoustic cloaks for pressure waves [15,16]. Such cloaks require an anisotropic mass density.

In the present Letter, we show that it is possible to design a cylindrical acoustic cloak for linear surface liquid waves. We constructed a structured metamaterial which bends surface waves over a finite interval of Hertz frequencies. We demonstrate theoretically its mechanism using homogenization theory: for wavelengths which are large compared with its typical heterogeneity size, the cloak behaves as an effective fluid characterized by a transversely anisotropic shear viscosity.

Let Ω denote the region of the vessel occupied by the fluid. The conservation of momentum leads to the Navier-Stokes equations:

$$\rho\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\mathbf{u} - \mu\nabla^2\mathbf{u} = -\nabla p + \rho\mathbf{g}, \quad \text{in } \Omega, \qquad (1)$$

where $\mathbf{u}$ is the velocity field, $\mu\nabla^2\mathbf{u}$ accounts for the fluid's viscosity, p the fluid pressure, ρ its density and $\mathbf{g}$ the vector of gravity force. Note that $\mathbf{g} = -g\mathbf{e}_3$, where g denotes the acceleration caused by gravity and $\mathbf{e}_3$ a vertical unit vector.

In our experimental setup, we choose a fluid which is inviscid, so that $\mu\nabla^2\mathbf{u}$ can be neglected outside the cloak. The fluid should be also incompressible (divergence free) irrotational (curl free) and undergo only small fluctuations around a mean vertical position $x_3 = h$ (interface air-liquid), the bottom of the vessel being at $x_3 = 0$. Under all these assumptions, we deduce that the vertical displacement of the liquid ξ is related to a potential ϕ through $\xi(r, \theta, t) = \Re e(-\frac{i\omega}{g}\phi(r, \theta)e^{-i\omega t})$, where ω is the wave frequency and ϕ satisfies Helmholtz's equation [17]

$$\nabla^2\phi + \kappa^2\phi = 0, \quad (2)$$

on the free surface. This equation is valid in the liquid region outside the rigid cylinders and is supplied with a no-flow condition $\frac{\partial\phi}{\partial n} = 0$ on each cylinder's boundary. Furthermore, κ is a spectral parameter linked to the wave frequency via the dispersion relation [17]

$$\omega^2 = g\kappa(1 + \kappa^2 d_c^2)\tanh(\kappa h), \quad (3)$$

with d_c the liquid capillarity length. Note that surface waves propagating at the liquid-air interface are thus always dispersive, unlike pressure waves within the liquid.

Our aim is to homogenize the structured cloak in order to obtain an effective anisotropic fluid described by a viscosity matrix with a large enough entry in the θ direction: in that case, the liquid will flow faster in the azimuthal direction, and will therefore be bent around the central region of the cloak. To show this, we use a two-scale expansion method [18].

When the fluid penetrates the structured cloak Ω_c, it undergoes fast periodic oscillations: Ω_c $(R_1 \leq r \leq R_2, 0 \leq \theta < 2\pi)$ is evenly divided into a large number of small curved sectors ηY of radial length $\eta(R_2 - R_1)$ and azimuthal length $2\pi\eta$, where η is a small positive real parameter. The smaller η, the larger the number of small sectors ηY. The homogenization technique amounts to looking at the limit when η goes to zero, while the spectral parameter κ in Eq. (2) remains fixed (so is the wave frequency). To filter these oscillations, we consider an asymptotic expansion of the potential field solution of the Helmholtz Eq. (2) in terms of a macroscopic (slow) variable $\mathbf{x} = (r, \theta)$ and a microscopic (fast) variable $\mathbf{x}/\eta$

$$\forall\ \mathbf{x} \in \Omega_c, \qquad \phi_\eta(\mathbf{x}) = \sum_{i=0}^{\infty} \eta^i \phi^{(i)}(\mathbf{x}, \mathbf{x}/\eta), \quad (4)$$

where $\phi^{(i)}(\mathbf{x}, \cdot)$ is Y-periodic.

Rescaling the differential operator in Eq. (2) accordingly as $\nabla = \nabla_{\mathbf{x}} + \frac{1}{\eta}\nabla_{\mathbf{y}}$, and collecting terms of same powers of η, we obtain the following homogenized problem in the limit when η tends to zero (see also [18]):

$$\nabla \cdot ([\mu_{\mathrm{hom}}]\nabla\phi_{\mathrm{hom}}(\mathbf{x})) = \kappa^2\phi_{\mathrm{hom}}(\mathbf{x}) \quad \text{in } \Omega_c. \quad (5)$$

Our result shows that the velocity field is now solution of Eq. (1) with an anisotropic matrix of viscosity whose nontrivial part (transverse shear) is given by

$$[\mu_{\mathrm{hom}}] = \frac{1}{\mathcal{A}(Y^*)}\begin{pmatrix} \mathcal{A}(Y^*) - \psi_{rr} & \psi_{r\theta} \\ \psi_{\theta r} & \mathcal{A}(Y^*) - \psi_{\theta\theta} \end{pmatrix}. \quad (6)$$

Here, $\mathcal{A}(Y^*)$ denotes the area of the region Y^* surrounding a rigid inclusion (subject to Neumann boundary conditions) in an elementary cell Y of the periodic array, and ψ_{ij} represent corrective terms

$$\forall\ i, j \in \{r, \theta\}, \qquad \psi_{ij} = -\int_{\partial S} \Psi_i n_j ds, \quad (7)$$

where $\mathbf{n}$ is the unit outward normal to the boundary ∂S of the rigid inclusion in the cell Y.

Furthermore, Ψ_j, $j \in \{r, \theta\}$, are periodic potentials which are unique solutions (up to an additive constant) of the following two Laplace equations $(\mathcal{L}_j)$:

$$(\mathcal{L}_j)\text{: } \nabla^2\Psi_j = 0 \quad \text{in } Y^*, \quad (8)$$

which are supplied with the effective boundary condition $\frac{\partial\Psi_j}{\partial n} = -\mathbf{n} \cdot \mathbf{e}_j$ on the boundary ∂S of the inclusion. Here, $\mathbf{e}_r$ and $\mathbf{e}_\theta$ denote the vectors of the basis in polar coordinates (r, θ).

Modeling of the invisibility cloak has been carried out using the commercial finite elements package COMSOL. Figure 1 shows a snapshot of the surface waves for both

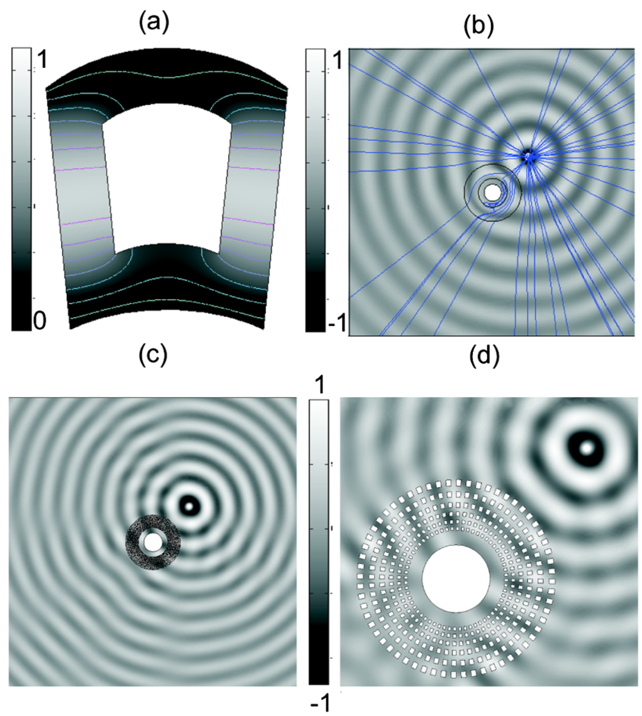

FIG. 1 (color online). (a) Periodic hydrostatic field Ψ_θ solution of Eq. (8) (red isovalues correspond to larger values of the potential); (b) Pattern of the concentric surface wave and associated stream lines (indicating the direction of fluid flow, in blue) generated by a forced term of spectral parameter $\kappa = 7$ [i.e., frequency 15.84 Hz for a depth of liquid $h = 9$ mm and a capillarity $d_c = 0.95$ mm using Eq. (3)] in presence of a rigid cylinder of radius 38 mm surrounded by a homogenized cloak [an artificial anisotropic fluid of shear viscosity given by Eq. (9)] lying in a ring with inner and outer radii $R_1 = 41$ mm and $R_2 = 100$ mm; (c) Same as (b) for a structured cloak with 256 curved sectors; (d) Structured cloak with 100 curved sectors and an acoustic source at frequency 9.81 Hz ($\kappa = 3.5$) as in the experiments for comparison with Fig. 3 right.

the homogenized (b) and the structured coatings, in the case of 256 sectors (c) and 100 sectors (d). A comparison between the ideal homogenized coating and the structured one shows that even if not perfect the structure actually works as expected. The fluid flowing within the homogenized cloak is characterized by the shear matrix

$$[\mu_{\text{hom}}] = \begin{pmatrix} 1.7 & 0 \\ 0 & 8.2 \end{pmatrix}. \tag{9}$$

In Fig. 1(a), we plot the periodic potential Ψ_θ used in the numerical computation of the second diagonal entry of the matrix $[\mu_{\text{hom}}]$ by solving Eq. (7) and (8) with $i = j = \theta$. It is evident from Fig. 1(a) that this potential takes larger values along the azimuthal direction, in agreement with the resulting anisotropy in (9).

Figure 1(b) clearly shows that the homogenized cloak associated with a structured cloak consisting of a very large number of identical curved sectors small compared to the working wavelength and regularly arranged along the r and θ directions, will enable one to gain control over surface waves. But we numerically checked that the cloaking is further improved by varying the size of sectors along the radial direction: it is enough to design a cloak with identical sectors to gain a good control of the velocity field's streamlines [Fig. 1(b)], but the azimuthal shear wave speed of liquid particles will increase linearly with their distance to the center of the cloak only in the case of sectors with increasing size. We numerically checked that in the case of a cloak with identical sectors, this shear wave speed does not vary; hence, a shadow region revealing the presence of the hidden object is observed behind the cloak (through phase shift).

Ideally, the effective fluid should be characterized by some varying density ρ, as well as a varying radial and azimuthal shear viscosities μ_{rr} and $\mu_{\theta\theta}$ [14]. These requirements seem to be out of reach through standard homogenization [18]. Nevertheless, we can introduce some variation in the radial length of sectors for which it seems reasonable to assume that the improved cloak is characterized by an effective anisotropic fluid whose shear viscosity (a diagonal matrix in polar basis) is

$$\mu'_{rr} = \left(\frac{R_2(r - R_1)}{(R_2 - R_1)r}\right)^2, \qquad \mu'_{\theta\theta} = \left(\frac{R_2}{R_2 - R_1}\right)^2, \tag{10}$$

where R_1 and R_2 are, respectively, the inner and outer radii of the ring. Importantly, the effective fluid's density $\rho' = \rho$; i.e., it does not play any prominent role. These parameters were actually first proposed in [8] for the case of electromagnetic waves. Nevertheless, homogenization theory requires some periodicity in both the r and θ directions [18]; hence, we are unable to derive rigorously this set of parameters. We numerically checked that such a structured cloak with 256 curved sectors [Fig. 1(c)] is more conducive for the cloaking than with 100 curved sectors [Fig. 1(d)]. But we had to find a compromise between the realizable structures using conventional machining, the limits imposed by the viscosity of the liquids, and the constraints imposed by homogenization.

The experimental setup is sketched in Fig. 2 (left). The liquid used for the experiments was methoxynonafluorobutane chosen for its physical properties and especially for having a low kinematic viscosity [17] $\nu = \mu/\rho = 0.61\ \text{mm}^2/s$) so that $\nu\nabla^2\mathbf{u}$ can be neglected outside the cloak in Eq. (1), a small surface tension $\sigma = 13.6$ N/cm and a large density ($\rho = 1.529$ g/mL), ensuring a small capillarity length $d_c = \sqrt{\sigma/(\rho g)} = 0.95$ mm. The vessel is filled with a depth of liquid $h = 9$ mm.

The basic principle behind the experiments is very simple: the light of a halogen lamp modulated by a perforated rotating disc illuminates a transparent vessel containing the liquid. The surface waves are excited by a localized pressure thanks to air pulsed in a small tube at the same frequency as the modulation of the light (to take advantage of the stroboscopic effect for the observation). The surface waves create local curvatures of the liquid and the light is refracted when crossing the surface. Thus, on the screen the dark and light zones allow visualizing the liquid surface waves. Note that the low viscosity of the liquid is important for such experiments and we were unable to produce similar results with water: due to its large viscosity, the water profile flattens within the microstructured cloak, much like in thin channels [17], and water cannot flow. Classical numerically controlled machine tools have been used to manufacture the invisibility cloak shown in Fig. 2 (right). The outer and inner radii are $R_2 = 100$ mm and $R_1 = 41$ mm, respectively. The cloak is divided in 100 identical angular sectors as shown in the picture and there are seven rows of rods along the radius. Snapshots of the

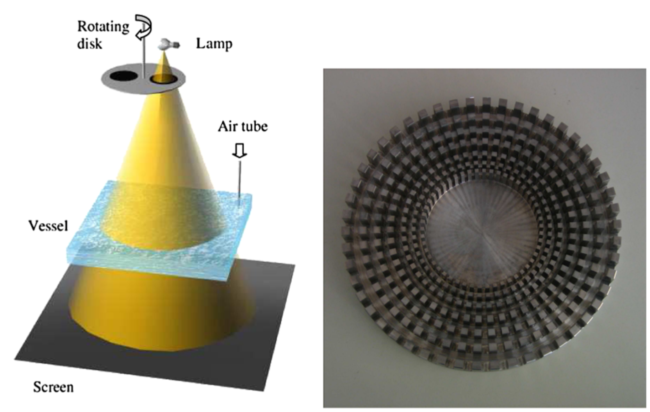

FIG. 2 (color online). Left (experimental setup): a halogen lamp modulated by a perforated rotating disc illuminates a transparent vessel containing the liquid (methoxynonafluorobutane). The surface waves are excited by a localized pressure thanks to air pulsed in a small tube at the same frequency as the modulation of the light (stroboscopic effect); Right (structured cloak): 100 rigid sectors are evenly machined in a metallic ring of inner radius $R_1 = 41$ mm and outer radius $R_2 = 100$ mm.

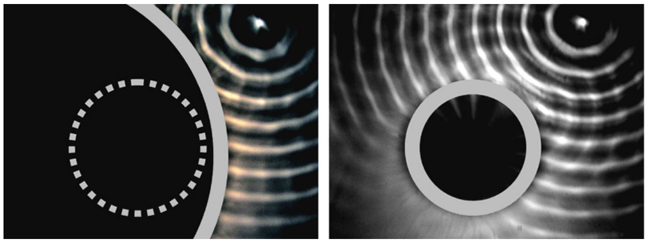

FIG. 3 (color online). Diffraction pattern of the surface waves (snapshot) generated by an acoustic source at frequency 10 Hz. The depth of liquid in the vessel is $h = 9$ mm and its capillarity is $d_c = 0.95$ mm. Left: diffraction by a rigid cylinder of radius 38 mm surrounded by the structured cloak (outlined as the gray coating) of Fig. 2 with inner and outer radii $R_1 = 41$ mm and $R_2 = 100$ mm. Right: diffraction by the rigid cylinder on its own (outlined by a dashed gray circle in the left panel for comparison).

liquid surface waves when a metallic cylinder is placed in the vessel alone and with the invisibility cloak are shown in Fig. 3. We emphasize that although four inclined sides in polymer foam were used to reduce the side effects (i.e., wave reflections at the vessel boundaries), the size of the vessel is such that it was not possible to observe a large area around the cloak; hence, no forward scattering measurements could be made. Nevertheless, the comparison of the velocity field patterns generated by the closely located acoustic source at 10 Hz in Fig. 3 right and left suggests that while the cylinder is much smaller and further away from the source than the invisibility cloak, it alters the wave pattern considerably more than the coat. Indeed, the wave backscattered by the cylinder is less apparent in Fig. 3 left than in Fig. 3 right.

We have proposed to use a structured metamaterial in order to create artificial anisotropic fluids bending the trajectory and accelerating the shear wave speed of particles at the surface of a liquid displaying a small azimuthal viscosity. Our theoretical model, based on homogenization theory, is in good agreement with full wave numerical solutions providing numerical evidence that our design of structured cloak works over a finite range of frequencies (spectral parameter κ ranging from 3.5 to 7) in the case of shallow Newtonian liquids. An experimental setup further suggests that control of liquid surface waves through artificial anisotropy is achieved in a vessel filled with 9 mm of Methoxynonafluorobutane around the frequency 10 Hz. Numerical simulations show that the cloaking is preserved at 15.84 Hz provided the structure of the cloak is refined (256 sectors instead of 100). A complete experimental proof of this broadband cloaking was beyond the scope of this letter, but we are confident that forward scattering measurements that would definitively support our claim will be reported in the near future. Interestingly, the viscosity of the liquid is less problematic for larger scales for which Eq. (3) brings no more genuine dispersion: our design could be used to protect off-shore platforms or coastlines from ocean waves such as tsunamis. Last, we would like to emphasize that our theoretical model can be used *mutatis mutandis* to design acoustic cloaks controlling antiplane shear waves in structured elastic material with cracks: for this take $\phi = u_3$, out-of-plane component of the displacement field, in Eq. (2) and assume that Eq. (3) reduces to $\omega = \kappa$. Such neutral inclusions [13] are characterized by an orthotropic homogenized coating of density $\rho = 1$ and shear modulus $[\mu_{\mathrm{hom}}]$ (a rank 2 tensor).

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