Consistency Restrictions on Maximal Electric-Field Strength in Quantum Field Theory

S. P. Gavrilov[1,*] and D. M. Gitman[2,+]

[1]*Department of General and Experimental Physics, Herzen State Pedagogical University of Russia, Moyka emb. 48, 191186 St. Petersburg, Russia*

[2]*Institute of Physics, University of São Paulo, CP 66318, CEP 05315-970 São Paulo, SP, Brazil*

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Quantum field theory with an external background can be considered as a consistent model only if backreaction is relatively small with respect to the background. To find the corresponding consistency restrictions on an external electric field and its duration in QED and QCD, we analyze the mean-energy density of quantized fields for an arbitrary constant electric field E, acting during a large but finite time T. Using the corresponding asymptotics with respect to the dimensionless parameter eET^2, one can see that the leading contributions to the energy are due to the creation of particles by the electric field. Assuming that these contributions are small in comparison with the energy density of the electric background, we establish the above-mentioned restrictions, which determine, in fact, the time scales from above of depletion of an electric field due to the backreaction.

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It is well known that QFT in an external background provides an efficient model for the study of quantum processes in those cases when some part of a quantized field is strong enough to be treated as a classical one. For example, QED with an external electromagnetic background formally arises from extending the QED Lagrangian by the interaction of the matter current with a given external electromagnetic field A_μ^{ext}, which is not quantized. This is naturally implied as a certain approximation. The study of some problems in QED and QCD with superstrong external backgrounds and their applications to astrophysics and condensed matter (to graphene physics) has once again raised the question of a consistency of such theories. Obviously, the question must be answered, first of all, in the case of a constant external field. Calculations that have an immediate relation to the above-mentioned problem have first been carried out by Heisenberg and Euler in the case of QED with constant parallel electric E and magnetic B fields, see [1]. They computed the change of the vacuum energy of spinning particles for arbitrary B and a weak electric field $E \ll E_c = m^2/e$ ($\hbar = c = 1$) which is unable to effectively create pairs from vacuum. They interpreted this change as a change of the energy of the external field itself and, at the same time, as a change of the Maxwell Lagrangian $\mathcal{L}^{(0)} = (E^2 - B^2)/8\pi$ by a certain addition $\mathcal{L}^{(1)}$. The limiting case of a strong magnetic field ($B \gg m^2/e$, $E = 0$) yields

$$\mathcal{L}^{(1)} \approx -\left(\frac{\alpha}{3\pi}\ln\frac{eB}{m^2}\right)\mathcal{L}^{(0)}, \tag{1}$$

where $\alpha = e^2$ is the fine-structure constant. This result is in agreement with more advanced calculations carried out by Ritus [2], who arrived at the conclusion that the loop expansion makes sense only for the magnetic fields restricted by the condition $B \ll F_{\max}$,

$$F_{\max} = \frac{m^2}{e}\exp\left(\frac{3\pi}{\alpha}\right) \approx \frac{m^2}{e}10^{560}.$$

Shabad and Usov [3] have recently established a more rigid limitation for the maximal admissible strength of magnetic field, $B \ll 10^{28}m^2/e$, having analyzed the structure of QED of vacuum in this field, taking into account the interaction of virtual electron-positron pairs.

The addition $\mathcal{L}^{(1)}$ has been generalized in a certain way to an arbitrary constant field (to arbitrary E) by Schwinger [5] and is now called the Heisenberg-Euler Lagrangian (HEL); for a review, see [4]. However, its physical meaning for a strong electric field is not completely clear. In the general case, the HEL is complex valued, its imaginary part determines the probability of pair-creation, as has been confirmed by independent calculations [6,7]. Considering the in-out vacuum current, Schwinger has made it possible to obtain an elegant expression for his effective Lagrangian in terms of the causal (Feynman's) Green function. Nevertheless, such an effective Lagrangian is not related to the problem of mean values, in particular, it does not reproduce the change of the vacuum energy of spinning particles for arbitrary E; the latter problem has to be formulated independently as a mean-value problem, and is expressed via noncausal Green's functions; see [7].

In a strong electric field ($E \gg m^2/e$, $B = 0$), the real-valued part of HEL, describing the effects of vacuum polarization, is given by the right-hand side of (1), where B is replaced by E. From this expression one can extract the negative-valued additive correction $\mathcal{E}^{(1)} = \text{Re}\mathcal{L}^{(1)}$ to the classical Maxwell density of energy, $\mathcal{E}^{(0)} = \mathcal{L}^{(0)}$. On these grounds, in [8] it was suggested that $F_{\max}$ should be also a limiting value of electric field. It turns out that $\text{Im}L^{(1)} \sim E^2$, which can be interpreted as an evidences that the vacuum instability is less than the vacuum polarization. This is not true since the vacuum instability is a nonlocal effect, being directly dependent on the electric field duration.

In our opinion, the most adequate object, whose analysis can answer these questions, is the mean-energy density of matter for large values of strength and duration of the electric field, computed with respect to various initial states when the initial state is vacuum, and is in thermal equilibrium. A detailed calculation of such a mean value in QED in the one-loop approximation, taking an exact account of the interaction with the electric background, has been given in [9]. In addition, we consider the case of charged bosons and QCD with an external chromoelectric field. We demonstrate that under these conditions, the effect of particle-creation is precisely the main reason for the change of the energy of matter.

A constant electric field acting during an infinite time creates an infinite number of pairs from vacuum even in a finite volume. This is why we choose the external background as a constant electric field acting during a finite period of time; we refer to this field as a T-constant field. The finiteness of field duration is a natural regularization in the given problem; on the other hand, it is a necessary physical parameter, which subsequently enters the consistency restriction on the value of the maximal electric field. The T-constant field turns on at $-T/2 = t_1 > t_{\rm in}$ and turns off at $T/2 = t_2 < t_{\rm out}$. We choose the nonzero T-constant field potential $A_3(t)$ as a continuous function of the form $A_3(t) = -Et$ for $t \in [t_1, t_2]$, being constant for $t \in (-\infty, t_1)$ and $t \in (t_2, +\infty)$. The effects of particle creation by the T-constant field have been studied in detail in [10]. In particular, it was shown that in case

$$T \gg (eE)^{-1/2}[1 + m^2/eE] \tag{2}$$

all the finite effects caused by particle-creation reach their asymptotic values, whereas the details involving the form of switching the field on and off can be neglected. In our calculations, we assume this restriction from below for the time T.

Taking the time instant $t_2 - 0 = T/2 - 0$ immediately before the electric field has been turned off, we shall now examine the mean-energy value $\langle \hat{H} \rangle$ of the spinor field on condition that its state at the initial instant $t_{\rm in} \to -\infty$ should be vacuum. The potentials of the T-constant field do not depend on the spatial coordinates, which implies that $\langle \hat{H} \rangle$ is proportional to the space volume V. In the one-loop approximation, $\langle \hat{H} \rangle = wV$, with the mean-energy density w being independent of the spatial coordinates,

$$w = \tfrac{1}{2}\langle 0, \mathrm{in}|[\psi(x)^\dagger, \mathcal{H}\psi(x)]|0, \mathrm{in}\rangle|_{x^0 = t_2 - 0}, \tag{3}$$

where $\mathcal{H}$ is the one-particle Dirac Hamiltonian; $\psi(x)$ are the operators of the Dirac field in the generalized Furry representation (see, e.g., [7]) obeying the Dirac equation with the external background; $|0, \mathrm{in}\rangle$ is the initial vacuum state in the same representation. The above-mentioned choice allows one to take a complete account of the pair-creation effect during the entire time. Also, since the electric field has not yet been switched off, this allows us to make a complete study of the vacuum polarization effect. Notice that the initial vacuum $|0, \mathrm{in}\rangle$ is identical with the vacuum of those free particles that correspond to the initial potential $A_3 = ET/2$.

The expression for w is obviously real valued. One can see that it can be represented as

$$w = -\tfrac{1}{4}\Big(\lim_{t \to t'-0} \mathrm{tr}[(\partial_0 - \partial'_0)S_{\rm in}(x, x')] + \lim_{t \to t'+0} \mathrm{tr}[(\partial_0 - \partial'_0)S_{\rm in}(x, x')]\Big)\Big|_{\mathbf{x}=\mathbf{x}', x^0 = t_2 - 0}, \tag{4}$$

where $\mathrm{tr}[\cdots]$ is the trace in the space of 4×4 matrices, and $S_{\rm in}(x, x')$ is the so-called in-in Green function,

$$S_{\rm in}(x, x') = i\langle 0, \mathrm{in}|T\psi(x)\bar{\psi}(x')|0, \mathrm{in}\rangle = S^c(x, x') + S^p(x, x'), \tag{5}$$

where $S^c(x, x')$ is Feynman's causal Green function, while the function $S^p(x, x')$ is a difference of two Green's functions, satisfying the homogeneous Dirac equation; see [7]. The final vacuum $|0, \mathrm{out}\rangle$ is the vacuum of free particles in the generalized Furry picture and corresponds to the constant potential $A_3 = -ET/2$.

The separation of $S_{\rm in}$ into the c- and p-parts is responsible for the separation of w into the two respective summands $w = w^c + w^p$. One can verify that the expression for w^c has a finite limit at $T \to \infty$; i.e., it permits a transition to the limit of a constant electric field. Then S^c can be presented by a proper-time integral; see [11]. Using this expression, one can readily verify that w^c is expressed in terms of the real-valued part of HEL (at $B = 0$). This contribution is due to vacuum polarization. In a superstrong electric field, it has the form

$$w^c = E\frac{\partial \mathrm{Re}\mathcal{L}^{(1)}}{\partial E} - \mathrm{Re}\mathcal{L}^{(1)} \approx -\left(\frac{\alpha}{3\pi}\ln\frac{eE}{m^2}\right)\mathcal{L}^{(0)}.$$

The contribution w^p arises due to particle-creation. It is computed as follows. First of all, using the general theory of particle-creation (see, [7]), one can represent the function $S^p(x, x')$ in the form

$$S^p(x, x') = i\sum_{nm} {}_{-}\psi_n(x)[G({}_{+}|^{-})G({}_{-}|^{-})^{-1}]^\dagger_{nm} {}_{+}\bar{\psi}_m(x'). \tag{6}$$

Here, $\{{}_{\pm}\psi_n(x)\}$ are the so-called in-solutions of the Dirac equation in a T-constant electric field, their asymptotics at $t \le t_1$ being stationary states of free electrons $(+)$ and positrons $(-)$ for the Dirac Hamiltonian with the constant potential $A_3 = ET/2$. The matrices $G({}_{\pm}|^{\pm})$ (being a matrix generalization of the Bogolyubov coefficients) are defined by decompositions of the so-called out-solutions in the in-solutions: ${}^{\pm}\psi(x) = {}_{+}\psi(x)G({}_{+}|^{\pm}) + {}_{-}\psi(x)G({}_{-}|^{\pm})$. Here, $\{{}^{\pm}\psi_n(x)\}$ are the out-solutions of the Dirac equation in a

T-constant electric field. The quantum numbers of particles are chosen as $n = (\mathbf{p}, r)$, where $\mathbf{p}$ is the particle momentum and $r = \pm 1$ is the spin projection; the asymptotics of the out-solutions at $t \geq t_2$ describe free particles [electrons $(+)$ and positrons $(-)$] with an energy spectrum defined by the Dirac Hamiltonian with the constant potential $A_3 = -ET/2$ as follows $\varepsilon_{\mathbf{p},r} = \sqrt{m^2 + \mathbf{p}_\perp^2 + (eET/2 - p_3)^2}$, $\mathbf{p}_\perp = (p^1, p^2, 0)$. The matrices $G({}_\pm|^\pm)$ are diagonal and can be expressed via the differential mean number of pairs created from vacuum, $\aleph_{\mathbf{p},r}$. Then we obtain

$$w^p = \frac{1}{4\pi^3} \int d\mathbf{p} \sum_{r=\pm 1} \aleph_{\mathbf{p},r} \varepsilon_{\mathbf{p},r}. \tag{7}$$

This quantity is the mean-energy density of pairs created from vacuum. It can be estimated in the case of strong electric fields, $E \gtrsim E_c$ and a sufficiently large T as follows. As has been demonstrated in [10], in case the time T is sufficiently large,

$$T \gg (m^2 + \mathbf{p}_\perp^2 + eE)(eE)^{-3/2},$$

and the longitudinal momenta are restricted by the condition $|p_3| \leq (\sqrt{eE}T/2 - K_p)\sqrt{eE}$, where K_p is a sufficiently large arbitrary constant, $\sqrt{eE}T \gg K_p \gg 1 + (m^2 + \mathbf{p}_\perp^2)/eE$, the differential mean numbers $\aleph_{\mathbf{p},r}$ have the form

$$\aleph_{\mathbf{p},r}^{\text{asy}} = \exp\left(-\pi \frac{m^2 + \mathbf{p}_\perp^2}{eE}\right).$$

For any fixed $\mathbf{p}_\perp^2$, the function $\aleph_{\mathbf{p},r}$ is fast decreasing for $|p_3| > (\sqrt{eE}T/2 - K_p)\sqrt{eE}$. For this reason, we can disregard the contribution to the integral (7) due to the integration over such momenta p_3 in comparison with the main contribution, which is defined by the dimensionless parameter eET^2. The latter parameter, in fact, determines a large integration domain over p_3. In its turn, the exponential decrease of $\aleph_{\mathbf{p},r}^{\text{asy}}$ with the grows of $\mathbf{p}_\perp^2$ allows one to ignore the contributions to the integral (7) due to a large $\mathbf{p}_\perp^2/eE \gtrsim \sqrt{eE}T$. Consequently, in order to evaluate the term which leads in $\sqrt{eE}T$ in integral (7) we can replace $\aleph_{\mathbf{p},r}$ by $\aleph_{\mathbf{p},r}^{\text{asy}}$ under condition (2), while restricting the limits of integration over momenta by the region $|p_3| \leq \sqrt{eE}(\sqrt{eE}T/2 - K)$, where K is a sufficiently large arbitrary constant, $\sqrt{eE}T \gg K \gg 1 + m^2/eE$. Having calculated the integral (7) over $\mathbf{p}_\perp$, we obtain the T-leading term in the form

$$w^p = eET\aleph, \qquad \aleph = \frac{e^2E^2T}{4\pi^3} \exp\left(-\pi \frac{m^2}{eE}\right). \tag{8}$$

We now suppose that the energy density of particles $w = w^p$, arising precisely due to the action of a T-constant electric field, should be essentially smaller than the density of the electric field itself, being equal to the classical Maxwell density of energy, $\mathcal{E}^{(0)} = E^2/8\pi$. Thus, the condition of a smallness of backreaction is $w^p \ll E^2/8\pi$, which, owing to (8), takes the form of a restriction from above on the dimensionless parameter eET^2:

$$eET^2 \ll \frac{\pi^2}{2e^2} \exp\left(\pi \frac{m^2}{eE}\right). \tag{9}$$

On the other hand, all the asymptotic formulas have been obtained under condition (2), which restricts the mentioned parameter from below, $[1 + m^2/eE]^2 \ll eET^2$. Since $\pi^2/2e^2 \gg 1$, there exists a region of values of E and T that satisfies both the inequalities. We note that time scale from above in (9) is more restrictive than the scale derived from the rate of pair production; see [12].

In case the initial state is in thermal equilibrium at temperature θ, the mean-energy density has an additional term w_θ^c, which represents, in fact, the work of a T-constant field on particles from the initial state, as well as the term w_θ^p

$$w_\theta^p = -\frac{1}{4\pi^3} \int d\mathbf{p} \sum_{r=\pm 1} \aleph_{\mathbf{p},r} n_{\mathbf{p},r}(\text{in}) \varepsilon_{\mathbf{p},r},$$
$$n_{\mathbf{p},r}(\text{in}) = [\exp(\tilde{\varepsilon}_{\mathbf{p},r}/\theta) + 1]^{-1},$$

where $\tilde{\varepsilon}_{\mathbf{p},r} = \sqrt{m^2 + \mathbf{p}_\perp^2 + (qET/2 + p_3)^2}$ is the energy of a free in-particle. The latter term determines a temperature-dependent correction to the energy of particles created from vacuum; see [9]. The energies of particles that contribute to w_θ^c in the limit of a large T are mostly determined by a large longitudinal kinetic momentum, with the energy being of order eET, as well as the energies of particles created at $\theta = 0$ in the expression (8) for w^p. Given that, however, the density of initial particles is constant, being determined only by the initial condition, whereas the density of created particles increases in proportion with T. Therefore, at large T and E, w_θ^c can be neglected in comparison with w^p, and $w \simeq w^p + w_\theta^p$.

In case the initial state is in thermal equilibrium at low temperatures $\theta \ll eET$, the contribution w_θ^p turns out to be small in comparison with w^p. At high temperatures $\theta \gg eET$, the energy density has the form $w = (eET/6\theta)w^p$. Thus, the restriction (9) is valid both for the vacuum initial state and for a low-temperature initial thermal state. At high temperatures we have a weaker restriction:

$$\frac{(eE)^2T^3}{\theta} \ll \frac{3\pi^2}{e^2} \exp\left(\pi \frac{m^2}{eE}\right). \tag{10}$$

Analogously, one can find restrictions for QED with charged bosons in a T-constant electric field. At low temperatures, we have

$$eET^2 \ll \frac{\pi^2}{Je^2} \exp\left(\pi \frac{m^2}{eE}\right),$$

where J is the number of the spin degrees of freedom ($J = 1$ for scalar particles and $J = 3$ for vector particles). In the case of high temperatures, the restriction has a completely different character than (10), namely,

$$\theta T \ln(\sqrt{eE}T) \ll \frac{\pi^2}{2Je^2} \exp\left(\pi \frac{m^2}{eE}\right).$$

One can easily extend these results to $D + 1$ dimensions, using the corresponding N in (8), taken from Eq. (37) in [10].

A similar analysis can be performed in the case of QCD with an electriclike non-Abelian external background. Such a background is a part of the known chromoelectric flux-tube model [13]. At present, the chromoelectric field is associated [14] with an effective theory, color glass condensate. Here, we shall derive restrictions on the external background which allows one to treat particles created from vacuum still as weakly coupled, owing to the property of asymptotic freedom in QCD. To this end, we use the results obtained in [15,16] for QCD with a constant $SU(3)$ chromoelectric field E^a ($a = 1, \ldots, 8$). If the initial state is vacuum, the density of created gluons is noticeably higher than the density of created quarks at any intensity of a T-constant chromoelectric field; see [15]. The same is valid at any finite temperature; therefore, for our purposes it is sufficient to take into account only the gluon contribution. It has been demonstrated in [15] that the $\mathbf{p}_\perp$-distribution density $n^{\text{gluon}}_{\mathbf{p}_\perp}$ of gluons produced from vacuum with all the possible values p_3 and the quantum numbers that characterize the inner degrees of freedom can be presented as follows:

$$n^{\text{gluon}}_{\mathbf{p}_\perp} = \frac{1}{4\pi^3} \sum_{j=1}^{3} T q \tilde{E}_{(j)} \aleph^{(j)}_{\mathbf{p}}, \quad \aleph^{(j)}_{\mathbf{p}} = \exp\left(-\frac{\pi \mathbf{p}_\perp^2}{q\tilde{E}_{(j)}}\right), \quad (11)$$

where $\tilde{E}_{(j)}$ are positive eigenvalues of the matrix $if^{abc}E^c$ for the adjoint representation of $SU(3)$, and q is the coupling constant. The (j) terms in (11) can be interpreted as those obtained for Abelian-like electric fields $\tilde{E}_{(j)}$, respectively. Then, the total energy density of gluons created from vacuum by the field $\tilde{E}_{(j)}$ is determined by integrals of the kind (7). Taking into account that maxima of the fields are restricted by the condition $\tilde{E}_{(j)} \leq \sqrt{C_1}$ ($C_1 = E^a E^a$ is a Casimir invariant for $SU(3)$) and the relation $\sum_{j=1}^{3} \tilde{E}^2_{(j)} = 3C_1/2$, one can find that at low temperatures $\theta \ll q\sqrt{C_1}T$ the consistency restriction for the dimensionless parameter $q\sqrt{C_1}T^2$ has the form

$$q\sqrt{C_1}T^2 \ll \pi^2/3q^2.$$

As in the case of QED, this restriction must be accompanied by a restriction from below, $1 \ll q\sqrt{C_1}T^2$, which is related to the fact that all the asymptotic expressions have been obtained for sufficiently large values of T. Therefore, the T-constant $SU(3)$ chromoelectric field approximation is consistent during the period when the produced partons can be treated as weakly coupled, due to the property of asymptotic freedom in QCD. At high temperatures, $\theta \gg q\sqrt{C_1}T$, the consistency restriction is far more rigid:

$$\theta T \ln(q\sqrt{C_1}T^2) \ll \pi^2/3q^2.$$

The above established consistency restrictions determine, in fact, the time scales from above of depletion of an electric field due to the backreaction.

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*gavrilovsergeyp@yahoo.com
†gitman@dfn.if.usp.br

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