

## Creating Kinks from Particles

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We study the creation of solitons from particles, using the  $\lambda\phi^4$  model as a prototype. We consider the scattering of small, identical, wave pulses, which are equivalent to a sequence of particles, and find that kink-antikink pairs are created for a large region in parameter space. We also find that scattering at low velocities is favorable for creating solitons that have large energy compared to the mass of a particle.

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A wide variety of systems, ranging from polyacetylene and Josephson junctions to high energy particle physics models, contain nonperturbative, “soliton” or “solitary wave” excitations in addition to perturbative “particle” excitations [1]. An important unsolved problem is to find ways to transition from the particle sector to the soliton sector. At a pragmatic level, we would like to develop implementable schemes that might enable solitons to be built out of particles. A transition from two energetic particles to solitons, however, is known to be exponentially suppressed, e.g., [2] (for a review see [3]), though it may occur more readily in certain situations, such as in the background of a preexisting kink [4,5].

In this Letter, we will determine a class of initial conditions that consist of small amplitude perturbations that scatter and successfully lead to the production of a kink-antikink (“ $k\bar{k}$ ”) pair in  $1 + 1$  dimensions. A trivial scheme to determine such a set of initial conditions is to time reverse the annihilation of  $k\bar{k}$ . Then the time reversed particles would assemble into an outgoing  $k\bar{k}$ . However, in any practical setting, such initial conditions would be very hard to arrange since the characteristics of the radiation from  $k\bar{k}$  annihilation are highly nontrivial. Instead we want to consider “clean” initial conditions in which we scatter identical wave pulses, somewhat like 2 particle scattering. The simplicity of the initial state comes with a price in that the final state will now not only contain a  $k\bar{k}$  but also some radiation. Our approach thus differs from other studies which generally considered initial conditions containing a single kink and hence had nonvanishing topological charge, e.g., [5].

In order to determine clean initial conditions that give  $k\bar{k}$  in the final state, we draw lessons from the sine-Gordon model which contains both particle and soliton sectors and has been studied extensively, both classically and in quantum theory [1,6–8]. While the complete integrability of the

sine-Gordon model permits many exact solutions, it also leads to a disappointing disconnection between the particle and soliton sectors, not present in many other models which admit solitons. For example, in the sine-Gordon model, it is not possible to start with, say, a soliton and an antisoliton and end up with particles. If a soliton and an antisoliton are set up to collide and possibly annihilate, they simply pass through each other. Thus soliton scattering states do not convert to particle states (even in quantum theory).

What is important for us is that the sine-Gordon model also contains “breather states.” If a breather state has large amplitude, it can be interpreted as a bound state of a soliton and an antisoliton, in which the two keep oscillating about each other but never annihilate. On the other hand, small quantized breathers have been interpreted as fundamental particles in the theory. Then one might expect the breather to be a bridge between the particle and soliton sectors. In the sine-Gordon model, however, the breather is a stable object in itself and fails to connect the particle and soliton sectors.

To connect the particle and soliton sectors it is necessary to depart from the sine-Gordon model. The smaller the departure, the weaker will be the connection between the particle and soliton sectors. Then, if we depart weakly from the sine-Gordon model, we expect long-lived “breather-like” states that can transition to both widely separated kink-antikink pair and also to particles. Such long-lived states have been discovered in various systems and have been termed “bions” in certain contexts and “oscillons” in others [9–21].

Motivated by these considerations, we study  $k\bar{k}$  production in the  $\lambda\phi^4$  model

$$L = \phi_0^2 \left[ \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{4} (\phi^2 - 1)^2 \right], \quad (1)$$

where we have rescaled fields and coordinates so that  $\phi_0$  is the only parameter in the model. The equation of motion is

$$\ddot{\phi} = \phi'' - (\phi^2 - 1)\phi, \quad (2)$$

where overdots denote time derivatives and primes denote spatial derivatives. The mass of a fundamental excitation can be found by considering small fluctuations around one of the vacua (say  $\phi = +1$ ) and is  $m = \sqrt{2}$ . The kink profile is

$$\phi_k = \tanh\left(\frac{x}{\sqrt{2}}\right). \quad (3)$$

The energy of a kink is found from the energy expression

$$E = \phi_0^2 \int dx \left[ \frac{\dot{\phi}^2}{2} + \frac{\phi'^2}{2} + \frac{1}{4}(\phi^2 - 1)^2 \right] \quad (4)$$

and is

$$E_k = \frac{2\sqrt{2}}{3} \phi_0^2 = \frac{2m}{3} \phi_0^2. \quad (5)$$

Note that the kink energy may be made very large compared to the particle mass by taking large values of  $\phi_0$ . However,  $\phi_0$  itself does not enter the classical dynamics of the scalar field though it does play a role in the quantized model.

We would like to use breatherlike solutions in the  $\lambda\phi^4$  model in our initial condition. However, such solutions are not known analytically. Hence we simply use the breather solutions of the pure sine-Gordon model

$$L_{\text{SG}} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{\pi^2} [1 + \cos(\pi\phi)], \quad (6)$$

which are given by

$$\phi_b^{\text{SG}}(t, x; \omega, v) = -1 + \frac{4}{\pi} \tan^{-1} \left[ \frac{\eta \sin(\omega T)}{\cosh(\eta \omega X)} \right], \quad (7)$$

where

$$\begin{aligned} T &= \gamma[t - v(x - x_0)], & X &= \gamma[x - x_0 - vt] \\ \gamma &= (1 - v^2)^{-1/2}, & \eta &= \sqrt{1 - \omega^2}/\omega. \end{aligned} \quad (8)$$

In Eq. (7), the  $\tan^{-1}(\cdot)$  function is taken to lie in the interval  $(-\pi/2, +\pi/2)$ . Apart from the boost  $\gamma$  and shift  $x_0$ , a breather solution is labeled by the parameter  $\omega \in (0, 1)$ . The solution for small  $\omega$  can be viewed as a sine-Gordon kink and an antikink that are oscillating back and forth, merging and emerging forever. Note that the breather is localized around one vacuum (at  $\phi = -1$ ), and probes the second vacuum (at  $\phi = +1$ ) for durations that vary inversely with  $\omega$ . For  $\omega \approx 1$ , the breather describes oscillations in the vacuum around  $\phi = -1$ . In the quantum theory, these oscillations are quantized and the energy of the lowest quantum state is equal to that of a particle, leading to the identification of the lowest energy breather with the particle excitation in the model.

Before proceeding consider an initial unboosted sine-Gordon breather [Eq. (7) with  $v = 0$ ], in the  $\lambda\phi^4$  model with equation of motion given in Eq. (2). [That is, the initial condition is  $\phi(0, x) = \phi_b^{\text{SG}}(0, x; \omega, 0)$  and  $\dot{\phi}(0, x) = \dot{\phi}_b^{\text{SG}}(0, x; \omega, 0)$ .] The energy of the solution can be obtained by evaluating Eq. (4) at  $t = 0$  when  $\phi_b = -1$  for all  $x$ . Then the potential and gradient terms do not contribute, and the kinetic contribution is easily evaluated. As in the sine-Gordon model we find

$$E_b = \frac{16}{\pi^2} \sqrt{1 - \omega^2} \phi_0^2. \quad (9)$$

The ratio of kink to breather energy is

$$\frac{E_k}{E_b} = \frac{\pi^2}{12\sqrt{2}} \frac{1}{\sqrt{1 - \omega^2}} \approx \frac{\pi^2}{24} \frac{1}{\sqrt{1 - \omega}}, \quad (10)$$

where in the last expression we assume  $\omega \approx 1$ . The field profile itself,  $\phi(t, x)$ , can be obtained numerically and we have checked that it is oscillatory and long-lived. More specifically, we have shown that half of the initial energy  $E_b$  in the simulation box (itself much larger than the breather size) is radiated in a time  $T_{1/2} \approx 5 \times 10^4 \lambda^{-1.9}$ , independently of  $\omega$ .

We now turn to the problem at hand, namely the creation of  $k\bar{k}$  from particles. Our initial conditions will consist of a train of  $N_b$  little (i.e.,  $\omega \approx 1$ ) breathers coming in from the left and another identical train of  $N_b$  breathers coming in from the right. We will study the collision of these breather trains for a variety of parameters and look for the formation of  $k\bar{k}$ . Hence our initial condition corresponds to an incoming state

$$f(t, x) = -1 + \sum_{n=-N_b, \neq 0}^{N_b} \frac{4}{\pi} \tan^{-1} \left[ \frac{\eta \sin(\omega T_n)}{\cosh(\eta \omega X_n)} \right], \quad (11)$$

with

$$T_n = \gamma[t - v_n(x - x_{0n})], \quad X_n = \gamma[(x - x_{0n}) - v_n t], \quad (12)$$

where  $x_{0n} = a + nd$ ,  $v_n = -v < 0$  for  $n > 0$ , and  $x_{0n} = -a + nd$ ,  $v_n = +v > 0$  for  $n < 0$ . The parameter  $a$  is half the separation between the trains at  $t = 0$  and  $d$  is the separation between different breathers in the same train. The initial conditions (at  $t = 0$ ) are

$$\phi(0, x) = f(0, x), \quad \dot{\phi}(0, x) = \dot{f}(0, x). \quad (13)$$

To further motivate our choice of initial conditions, let us consider what might be required to form a  $k\bar{k}$ . Initially, the field is oscillating about the  $\phi = -1$  vacuum. To form  $k\bar{k}$ , we need the oscillations to extend into the  $\phi = +1$  vacuum. So we need to build up the field oscillations. As the leading breathers in the trains collide, the field at  $x = 0$  starts oscillating. Subsequent breathers provide additional kicks to the oscillations at  $x = 0$ . Provided the subsequent collisions are in phase, the amplitude of oscillations at

$x = 0$  will grow in resonance. The growth must compete with the dissipation due to the emission of particle radiation. If the growth wins, the oscillations will eventually extend up to the  $\phi = +1$  vacuum and then it will be energetically favorable for  $\phi(t, 0)$  to stay there. Then it becomes likely that  $k\bar{k}$  will be created.

The same heuristic argument may be applied to the pure sine-Gordon model and serves to show its limitations. We know that  $k\bar{k}$  are not created in the sine-Gordon model, but it is not because dissipation is stronger than resonant growth. Instead the integrability ensures that the breather trains pass unscathed through each other. So the heuristic argument should be taken as motivation but cannot be taken too literally; instead we must solve the equations of motion and check for  $k\bar{k}$  production. However, what seems clear is that  $k\bar{k}$  production may proceed via a resonance and, just as a child can swing higher and higher by timing her movement to within a factor of 2 per kick, this level of tuning may be all that is needed to produce  $k\bar{k}$ .

The equation of motion for the scalar field, Eq. (2), is solved numerically with the initial conditions in Eq. (11), using the iterated Crank-Nicholson method with two iterations [22], and absorbing boundary conditions [23] at the ends of the lattice. The fields are evolved for one light crossing time. As an additional check, we have also evolved the initial conditions using MATHEMATICA, though with fixed boundary conditions (see [24] for the notebook). The MATHEMATICA results are generally consistent with the Crank-Nicholson method, but there are a few discrepancies. These may be due to the different integration routines which also have different accuracies. The Crank-Nicholson implementation is more transparent and we find it more reliable, while the MATHEMATICA implementation is more convenient to use.

The problem contains many parameters, all related to the choice of initial condition:  $\omega$ ,  $v$ ,  $a$ ,  $d$ , and  $N_b$ . For a given value of  $\omega$ , Eq. (10) shows that, just on energetic grounds, we need  $N_b > 0.6/\sqrt{1-\omega^2}$ . We have taken  $N_b = \text{int}[2/\sqrt{1-\omega^2}] + 1$  where  $\text{int}[x]$  denotes the largest integer less than or equal to  $x$ . Somewhat arbitrarily, we take the initial half-separation of the trains to be  $a = 10/\eta\omega = 10/\sqrt{1-\omega^2}$ , corresponding to 10 (at rest) breather widths. The separation of the breathers in a train is taken to be  $d = 2/\sqrt{1-\omega^2}$ . The only parameters left to specify are  $\omega$  and  $v$ . For a breather to have energy comparable to the particle mass, and the kink to have energy much larger than the particle, we require  $\omega$  to be very close to 1. With  $\omega = 0.99$ , the kink energy is about 4 times that of a breather. We shall take  $\omega \in (0.90, 0.99)$ . We then do runs for different values of  $v$  and look for  $k\bar{k}$  formation.

An example of  $k\bar{k}$  production is shown in Fig. 1 where we give two snapshots of the evolution. Animations of the evolution may be found in Ref. [24]. Generally, by looking at the field profile, it is quite clear when a  $k\bar{k}$  has been

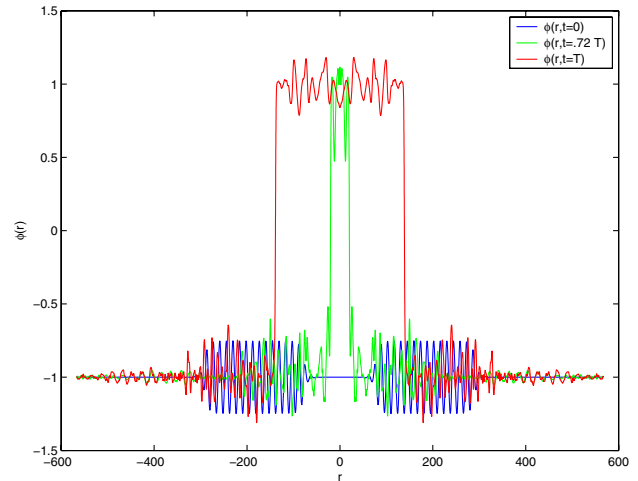


FIG. 1 (color online). Two snapshots of the collision of breather trains for  $\omega = 0.99$ ,  $v = 0.43$  and other parameters as described in the text.  $T$  denotes a light crossing time. The initial state contains the train of breathers. Subsequently, kinks appear and move apart.

created. However, there are some instances in which the outcome is not so clear-cut. This includes the case when the field profile shows  $k\bar{k}$  that are not separated by a large distance or are almost at rest with respect to each other. Then there is the possibility that the  $k\bar{k}$  will annihilate. In such cases, we have chosen to call it a  $k\bar{k}$  creation event if the kinks survived for at least the duration of the simulation. Another novel outcome we have seen is that for some parameters two or more pairs of  $k\bar{k}$  are produced.

In Fig. 2 we plot the region on the  $(\omega, v)$  plane for our choice of parameters that lead to the formation of a pair (or more) of  $k\bar{k}$ . Note the trend—higher  $\omega$ , i.e., smaller

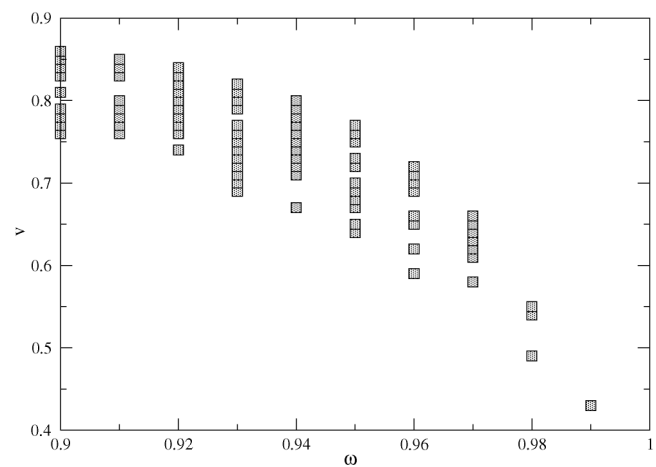


FIG. 2. Results from our Crank-Nicholson code (square symbols) mapping out  $k\bar{k}$  formation in the  $(\omega, v)$  plane for the choice of other parameters as described in the text. Note the occasional gaps where  $k\bar{k}$  are not formed, and the downward trend with larger  $\omega$  (weaker incoming pulses).

breather energy, requires lower incoming velocity. This indicates that it is preferable to scatter many particles at *low* energy to create solitons. If the incoming velocity is too high, the breather trains simply pass through, as in the sine-Gordon model. Also, note the occasional holes in the plot (e.g.,  $\omega = 0.91$ ,  $v = 0.82$ ) where we did not observe  $k\bar{k}$  formation. This substructure in the plot is reminiscent of the bands observed in  $k\bar{k}$  scattering [25] and suggests that  $k\bar{k}$  formation may be due to resonance.

The region leading to  $k\bar{k}$  formation is reasonably large but does not extend to arbitrarily high  $\omega$ . For example, we have not found initial conditions leading to  $k\bar{k}$  formation for  $\omega > 0.99$ . The expanse of the “successful” region does not concern us at the moment because our main objective was to find a set of clean initial conditions that led to the formation of  $k\bar{k}$ . We would be surprised if future investigations do not find a larger set of successful clean initial conditions, even for very high values of  $\omega$ . Whether these initial conditions are achievable in a practical setting is a separate matter, and depends on the details of the experiment.

There are several directions in which it would be useful to extend our results. The first is to scan the space of initial conditions more carefully, to gain further understanding of what conditions enable  $k\bar{k}$  formation. Our space of initial conditions could also be enlarged, if necessary. For example, different breathers in a train could come in with different velocities. We could also envision “building up” by starting with very large  $\omega$  (small energy) breathers, and building states corresponding to smaller  $\omega$  (larger energy), which can then collide to form  $k\bar{k}$ . Another direction is to include quantum effects in the scattering. This would require more precise understanding of the breather and kink states in terms of particles. In the quantum sine-Gordon model, soliton operators have been written down in terms of an infinite number of particle operators [26]. We expect that the soliton operator in the  $\lambda\phi^4$  model should be expressible in terms of a finite number of particle operators, otherwise it would seem impossible to build a  $k\bar{k}$  starting with particles. Yet another direction to proceed would be to consider solitons in higher dimensions. Then we can study the creation of vortex-antivortex or monopole-antimonopole pairs in suitable systems. We would clearly need higher dimensional analogs of breathers and we expect that oscillon states can play this role. Finally, it would be useful to generalize our initial state to real systems. After all, polyacetylene is described by the  $\lambda\phi^4$  model, and we may expect to be able to create  $k\bar{k}$  there. (Similar problems also arise in polymer physics in the context of polymers that pass through a membrane [27].) Our results do not directly apply to polyacetylene because the dynamics there is nonrelativistic. However,

with suitable generalization, it may become possible to test some of these ideas experimentally.

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- [1] T. Vachaspati, *Kinks and Domain Walls* (Cambridge University Press, Cambridge, England, 2006).
  - [2] D. G. Levkov and S. M. Sibiryakov, Phys. Rev. D **71**, 025001 (2005).
  - [3] M. P. Mattis, Phys. Rep. **214**, 159 (1992).
  - [4] N. S. Manton and H. Merabet, arXiv:hep-th/9605038.
  - [5] T. Romanczukiewicz, J. Phys. A **39**, 3479 (2006).
  - [6] R. Rajaraman, *Solitons and Instantons* (North-Holland, Amsterdam, 1987).
  - [7] S. Coleman, *Aspects of Symmetry: Selected Erice Lectures* (Cambridge University Press, Cambridge, England, 1985).
  - [8] R. F. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D **11**, 3424 (1975); **10**, 4130 (1974); **10**, 4114 (1974).
  - [9] I. L. Bogolyubsky and V. G. Makhankov, JETP Lett. **24**, 12 (1976).
  - [10] J. P. Boyd, Nonlinearity **3**, 177 (1990).
  - [11] M. Gleiser, Phys. Rev. D **49**, 2978 (1994).
  - [12] E. J. Copeland, M. Gleiser, and H. R. Muller, Phys. Rev. D **52**, 1920 (1995).
  - [13] J. P. Boyd, Wave Motion **21**, 311 (1995).
  - [14] E. P. Honda, arXiv:hep-ph/0009104.
  - [15] E. Farhi, N. Graham, V. Khemani, R. Markov, and R. Rosales, Phys. Rev. D **72**, 101701 (2005).
  - [16] M. Hindmarsh and P. Salmi, Phys. Rev. D **74**, 105005 (2006).
  - [17] G. Fodor, P. Forgacs, P. Grandclement, and I. Racz, Phys. Rev. D **74**, 124003 (2006).
  - [18] P. M. Saffin and A. Tranberg, J. High Energy Phys. **01** (2007) 030.
  - [19] N. Graham, Phys. Rev. Lett. **98**, 101801 (2007).
  - [20] M. Gleiser and J. Thorarinsson, Phys. Rev. D **76**, 041701 (R) (2007).
  - [21] G. Fodor, P. Forgacs, Z. Horvath, and A. Lukacs, Phys. Rev. D **78**, 025003 (2008).
  - [22] S. A. Teukolsky, Phys. Rev. D **61**, 087501 (2000).
  - [23] K. D. Olum and J. J. Blanco-Pillado, Phys. Rev. Lett. **84**, 4288 (2000).
  - [24] <http://www.hep.vanderbilt.edu/~duttas/kkbar.html>
  - [25] P. Anninos, S. Oliveira, and R. A. Matzner, Phys. Rev. D **44**, 1147 (1991).
  - [26] S. Mandelstam, Phys. Rev. D **11**, 3026 (1975).
  - [27] K. L. Sebastian and A. K. R. Paul, Phys. Rev. E **62**, 927 (2000).