Statistics of the Work Done on a Quantum Critical System by Quenching a Control Parameter

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We study the statistics of the work done on a quantum critical system by quenching a control parameter in the Hamiltonian. We elucidate the relation between the probability distribution of the work and the Loschmidt echo, a quantity emerging usually in the context of dephasing. Using this connection we characterize the statistics of the work done on a quantum Ising chain by quenching locally or globally the transverse field. We show that for local quenches starting at criticality the probability distribution of the work displays an interesting edge singularity.

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A series of recent experiments with cold atomic gases [1,2] spurred new interest on the dynamics of quantum correlated systems. A number of fundamental issues on the nonequilibrium physics of quantum systems are being addressed, ranging from the relation between nonintegrability and thermalization [3], to the universality of defect production for adiabatic quenches across quantum critical points [4]. In this broad context, a paradigmatic example of experimental protocol is the instantaneous quench: an abrupt change, either global or local, of a control parameter g from some initial value g_0 to a final one g_1. Experimentally, it has been shown that the dynamics after such quenches may show intriguing features, such as collapse and revivals of the order parameter for quenches done through a quantum critical point [1,5], as well as the absence of thermalization in systems close to integrability [2,3].

Theoretically, the study of quantum quenches received considerable interest: after a series of classic works on the nonequilibrium dynamics of the quantum Ising model [6], recent investigations focused on characterizing the long time asymptotics of correlation functions [7,8], their behavior as compared to their thermal counterparts [3], and the universality emerging in the quench dynamics at a quantum critical point [9]. Partial information on the internal dynamics of the system can be obtained in a variety of ways. One may extract the way in which excitations propagate by looking at the time dependence of correlators after a quench [8,9]. More subtle information on the establishment of quantum correlations can be obtained by studying the dynamics of entanglement entropies [10]. The purpose of this Letter is to discuss a more basic way to characterize both the internal dynamics and the quench protocol itself by obtaining information on how far from equilibrium the system has been taken. This can be done by studying the statistics of a fundamental quantity: the work W done on the system by changing its parameters.

The main observation behind this proposal is that the quench protocol resembles a standard thermodynamic transformation. However, since a quench takes the system out of equilibrium, the work W, unlike in a quasistatic process, is characterized by a probability distribution $P(W)$ [11–13]. Below, we focus on the characteristic function of $P(W)$, defined as

$$G(t) = \int dW e^{iWt} P(W), \tag{1}$$

and study it for the prototypical example of a quantum critical system, the quantum Ising chain. We first elucidate a useful relation between $G(t)$ and the Loschmidt echo, a quantity emerging in various physical contexts, most notably the Fermi edge singularity [14], quantum chaos [15], and the physics of dephasing [16,17]. Using this connection and a combination of field theoretic tools, we compute exactly and analytically $G(t)$ for global and local quantum quenches. In both cases, we characterize the fluctuations of the work and their probability distribution. Interestingly, we show that for a local quench starting at the quantum critical point the function $P(W)$ displays an edge singularity.

Let us start by briefly discussing the relation between the characteristic function $G(t)$ and the Loschmidt echo. For a generic quench $H(g_0) \to H(g_1)$, the Loschmidt echo is defined as $\mathcal{L}(t) = |\mathcal{G}(t)|^2$, where the amplitude $\mathcal{G}$ is given by

$$\mathcal{G}(t) = \langle e^{iH(g_0)t} e^{-iH(g_1)t} \rangle. \tag{2}$$

Here $H(g_0)$ and $H(g_1)$ are the initial and the final Hamiltonian, respectively, and the average is taken with respect to the initial equilibrium density matrix $\rho_0 = \exp[-\beta H(g_0)]/Z$. The Loschmidt echo can be seen as a measure of the sensitivity of the system to the quench. The connection with $P(W)$ emerges by noticing that for a generic quench the characterization of the work done on the system requires two energy measurements: one before it and one after it [12,13]. If the results of such measurements are $\tilde{E}$ and E, the work done is then $W = E - \tilde{E}$. Hence if $|\Phi_n\rangle$ are the eigenstates of energy E_n of $H(g_1)$, we have that

$$P(W) = \sum_{n,m} \delta[W - (E_n - \tilde{E}_m)] |\langle \Phi_n | \Psi_m \rangle|^2 P_m, \tag{3}$$

where $|\Psi_m\rangle$ are the eigenstates of H_0 with energy $\tilde{E}_m$, and $P_m = \exp[-\beta\tilde{E}_m]/Z$. The characteristic function is then $G(t) = \sum_n e^{i(E_n-\tilde{E}_m)t}|\langle\Phi_n|\Psi_m\rangle|^2 P_m$, which is readily recognized to be the complex conjugate of the amplitude defining the Loschmidt echo $G(t) = [\mathcal{G}(t)]^*$. This equality is actually a special case of the generalized quantum Jarzynski equality [11,12] recently derived in Ref. [13] for problems in which g is taken from g_0 to g_1 in a finite time interval along a generic path $g(t)$. The Loschmidt echo $\mathcal{L}(t)$ can be, in principle, measured by studying the dephasing of an auxiliary two level system coupled to the system of interest [15–17]. In the same setup, the probability distribution $P(W)$ can be directly extracted from the absorbtion spectra associated with optical transitions in the auxiliary two level system [18].

For a global quench the work done is extensive. Therefore in the thermodynamic limit the probability distribution $P(w)$ of the work per unit volume $w = W/V$ will be a strongly peaked function, with fluctuations scaling as $1/\sqrt{V}$. This suggests that P is a nontrivial function only for small systems or for local quenches. Despite this fact, it is interesting and instructive to study the work statistics for a paradigmatic example: a zero temperature global quench of the transverse field in a quantum Ising chain [19]. The latter is defined by the Hamiltonian

$$H_0 = -J\sum_i \sigma_i^x\sigma_{i+1}^x + g\sigma_i^z, \tag{4}$$

where $\sigma_i^{x,z}$ are spin operators at site i, J is an overall energy scale (below we set $J = 1$), and g is the strength of the transverse field. The one-dimensional quantum Ising model is the prototypical, exactly solvable example of a quantum phase transition, with a quantum critical point at $g_c = 1$ separating two mutually dual gapped phases, a quantum paramagnetic one ($g > g_c$) and a ferromagnetic one ($g < g_c$).

Let us now consider a global change at time $t = 0$ of the transverse field from an initial value g_0 to a final one g_1. The analysis of the Loschmidt echo can be efficiently performed after a Jordan-Wigner transformation [19]. In the fermionic representation, the Hamiltonian Eq. (4) takes the simple form

$$H(g) = \sum_{k>0}[g - \cos(k)](c_k^\dagger c_k - c_{-k}c_{-k}^\dagger) + i\sin(k)(c_k^\dagger c_{-k}^\dagger - c_{-k}c_k), \tag{5}$$

where c_k are fermionic operators. The diagonal form $H = \sum_{k>0}E_k(\gamma_k^\dagger\gamma_k - \gamma_{-k}\gamma_{-k}^\dagger)$, with energies $E_k = \sqrt{[g-\cos(k)]^2 + \sin(k)^2}$, is achieved after a Bogoliubov rotation $c_k = u_k(g)\gamma_k - iv_k(g)\gamma_{-k}^\dagger$, $c_{-k}^\dagger = u_k(g)\gamma_{-k}^\dagger - iv_k(g)\gamma_k$. The coefficients are given by

$$u_k(g) = \cos(\theta_k(g)), \qquad v_k(g) = \sin(\theta_k(g)), \tag{6}$$

where $\tan(2\theta_k(g)) = \sin(k)/[g - \cos(k)]$. In this representation, the Loschmidt echo for the quantum Ising model following both a global and a local quench of g has been recently shown to the expressible in terms of matrix determinants, which were afterwards analyzed numerically [16,17]. Below, we compute analytically the Loschmidt echo employing field theoretic tools, which, in contrast to previous approaches, have the advantage of giving clear insight into the physics of the problem.

Our first task is to express the ground state $|\Psi_0\rangle$ of energy E_0 of the initial Hamiltonian $H(g_0)$ in terms of the eigenmodes γ_k diagonalizing $H(g_1)$. If we call η_k the eigenmodes of H_0, it is easy to see that $\eta_k = U_k\gamma_k - iV_k\gamma_{-k}^\dagger$, where

$$U_k = u_k(g_0)u_k(g_1) + v_k(g_0)v_k(g_1), \tag{7}$$

$$V_k = u_k(g_0)v_k(g_1) - v_k(g_0)u_k(g_1). \tag{8}$$

Hence the equation $\eta_k|\Psi_0\rangle = 0$, characterizing our initial state, can easily be solved giving

$$|\Psi_0\rangle = \frac{1}{\mathcal{N}}\exp\left[i\sum_{k>0}\frac{V_k}{U_k}\gamma_k^\dagger\gamma_{-k}^\dagger\right]|0\rangle, \tag{9}$$

where $\mathcal{N}$ is the normalization constant and $|0\rangle$ is the vacuum of the fermions γ_k. The structure of this state closely resembles that of integrable boundary states encountered in statistical field theory. In particular, the amplitude $\mathcal{G}$ is given by

$$\mathcal{G}(t) = e^{iE_0t}\langle\Psi_0|e^{-iH(g_1)t}|\Psi_0\rangle = \frac{e^{-i\delta Et}}{\mathcal{N}^2}\langle 0|e^{\sum_k B^*(k)\gamma_k\gamma_{-k}}e^{\sum_k B(k)e^{-2iE_kt}\gamma_{-k}^\dagger\gamma_k^\dagger}|0\rangle, \tag{10}$$

where $B(k) = -iV_k/U_k$ and $\delta E = E_1 - E_0$, where E_1 is the ground state energy of $H(g_1)$. Up to an irrelevant prefactor, Eq. (10) maps after a Wick rotation $it \to \tau$ onto the partition function of a two-dimensional classical Ising model constrained on a cylinder of height τ and with boundary conditions on the two ends described by $|\Psi_0\rangle$. Hence, using techniques originally developed for integrable boundary states [20] it is easy to obtain

$$\mathcal{G}(t) = e^{-i\delta Et}e^{L\int_0^\pi (dk/2\pi)\log[(1+|B(k)|^2e^{-2iE_kt})/1+|B(k)|^2)]}, \tag{11}$$

where L is the linear size of the chain.

We can now derive all the cumulants C_n of the probability distribution of the work per unit length $P(w)$ by expanding in power series $\log(G(t/L)) = \sum_{n=0}^{+\infty} C_n(it)^n/n!$. As expected, from Eq. (11) we obtain that, as L grows, $C_n \propto 1/L^{n-1}$, leading to the suppression of all fluctuations in the thermodynamic limit. In order to study the effects associated with the presence of the quantum critical point, let us focus on the dependence on g_0 and g_1 of the average work per unit length and of its variance. The first is in general given by $\langle w\rangle = \delta E/L + w_e$, where the excess work is $w_e \geq 0$, in agreement with standard thermodynamic relations. The latter and the variance have a particularly simple form if the final transverse field is

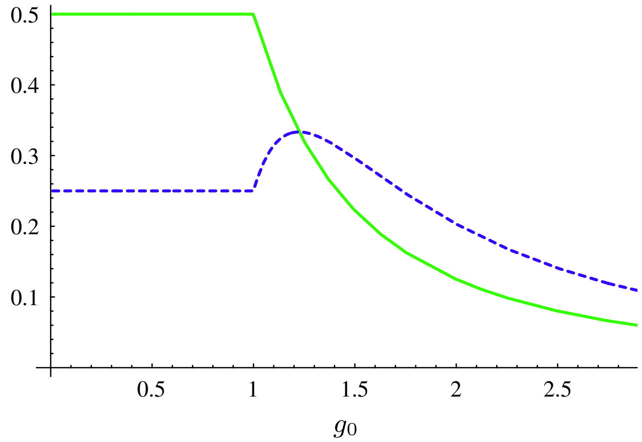

FIG. 1 (color online). A plot of the average excess work per unit length w_e/g_1 (solid line) and of the variance $L/g_1^2\langle(\Delta w)^2\rangle$ (dashed line) vs g_0 for a global quench from g_0 to $g_1 \gg 1$. The presence of the quantum critical point at $g_0 = 1$ is signaled by a discontinuity in the derivative.

large $g_1 \gg 1$, in which case $w_e = g_1(1 + g_0^2 - |g_0^2 - 1|)/4g_0^2$ and $\langle(\Delta w)^2\rangle = (g_1^2)/L\{g_0^4 + 4g_0^2 - 3 - \mathrm{sgn}[g_0^2 - 1](g_0^4 - 4g_0^2 + 3)\}/8g_0^4$. These functions are plotted in Fig. 1, where one may see that both w_e and $\langle(\Delta w)^2\rangle$ signal the presence of the quantum critical point with discontinuities in their derivative at $g = 1$. More striking universal effects associated with the quantum critical point are observed by studying the asymptotics of $\mathcal{G}(t)$ at long times, governed by the long wavelength modes. If, for example, one looks at $g_0, g_1 \neq 1$, one may easily obtain that

$$\mathcal{G}(t) \simeq \mathcal{G}^\infty \exp\left[\mathcal{A} m_1 L\left(\frac{m_1 - m_0}{m_1}\right)^2 \frac{e^{-2im_1 t}}{(imt)^{3/2}}\right], \quad (12)$$

where $\mathcal{G}^\infty$ is the asymptotic value attained by $\mathcal{G}$ [signaling the presence of a delta function peak in $P(w)$], $\mathcal{A}$ is a constant, and $m_i = |g_i - 1|$. Passing again to imaginary time, the dependence on t in the exponential corresponds to the dependence on the thickness of the free energy of the Ising model on a cylinder, which away from criticality is exponentially cutoff by the correlation length $\xi = 1/m_1$. At criticality, of course, it becomes a power law as a result of the establishment of long range correlations. A more detailed study of the statistics of $P(W)$ for global quenches will be reported elsewhere [18].

Let us now pass to a much more interesting situation in which we expect the work done on the system to show nontrivial fluctuations: a local quench of the Hamiltonian from $H_0 = H(g)$ to $H_0 + V$, where

$$V = -\delta g \sigma_0^z. \quad (13)$$

In order to capture the main differences with the previous case, let us start by considering the case $\delta g \ll 1$ and evaluate $\mathcal{G}(t)$ within a second-order cumulant expansion

$$\mathcal{G}(t) = \langle e^{iH_0 t} e^{-i(H_0+V)t}\rangle = \langle T e^{-i\int_0^t dt' V(t')}\rangle$$
$$= e^{-i\langle V\rangle t} e^{-(1/2)\int_0^t dt_1 dt_2 \langle T[V(t_1)V(t_2)]\rangle}, \quad (14)$$

where $V(t) = \exp[iH_0 t]V\exp[-iH_0 t]$. Using the fermionic representation of the spin operators, we obtain

$$V = \frac{\delta g}{L}\sum_{k,k'}[c_k c_{k'}^\dagger - c_k^\dagger c_{k'}]. \quad (15)$$

Hence writing V in terms of the eigenmodes η_k of H_0 and substituting in Eq. (14), with the help of Wick theorem, we obtain

$$\mathcal{G}(t) \simeq e^{-i\delta E t} e^{-f(t)}, \quad (16)$$

Here the energy shift δE is given by

$$\delta E = \delta g \int_{-\pi}^{\pi} \frac{dk}{2\pi}\cos(2\theta_k(g)) - \frac{(\delta g)^2}{2}\int_{-\pi}^{\pi}\frac{dk dk'}{(2\pi)^2} \times \frac{V(k,k')}{E_k + E_k'},$$

where $V(k,k') = \sin(2\theta_k)\sin(2\theta_{k'}) + 4\cos(\theta_k)^2\sin(\theta_k')^2$.

The most important information on the statistics of the work done on the system is contained in

$$f(t) = \frac{(\delta g)^2}{2}\int_{-\pi}^{\pi}\frac{dk dk'}{(2\pi)^2}\frac{V(k,k')}{(E_k + E_k')^2}(1 - e^{-i(E_k + E_{k'})t}). \quad (17)$$

From this expression we may again estimate the various cumulants of $P(W)$ by expanding in power series $f(t)$. In particular, the variance is given close to the critical point by

$$\langle(\Delta W)^2\rangle = \left(\frac{\delta g}{2\pi}\right)^2\left[2(1 + \pi^2) - (g - 1)\left(2 + \log\left[\frac{|g - 1|}{8}\right]\right)\right].$$

Despite the fact that this function has a logarithmic singularity of the first derivative at $g = 1$, as originally found in studies of dephasing [17], it is important to notice that the integral leading to this expression gets contributions from all frequencies (not just small k). Hence universality does not emerge substantially.

In order to obtain information on universal effects, one has to study the asymptotics of $\mathcal{G}(t)$ for long times. This can be done by looking at the asymptotic value attained by f at infinity $f_\infty = (\delta g)^2/2\int dk dk'/(2\pi)^2 V(k,k')/(E_k + E_{k'})^2$. Close to the quantum critical point $g \simeq 1$, this is given by

$$f_\infty \approx \left[\frac{\delta g}{2\pi}\right]^2 \log\left[\frac{1}{|g - 1|}\right]. \quad (18)$$

Hence as $t \to +\infty$ we have

$$\mathcal{L}(t) \simeq |g - 1|^{2(\delta g/2\pi)^2}. \quad (19)$$

The Loschmidt echo vanishes at the quantum critical point with a cusp singularity. As shown below, the vanishing of the Loschmidt echo is the result of an orthogonality catastrophe, originating from the changing of a local scattering potential in a nontrivial, yet gapless, effective fermionic system. In particular, if we set $g = 1$, the long time decay of $\mathcal{G}$ is a power law:

$$\mathcal{G}(t) \approx e^{-i\delta E t}(it)^{-(\delta g/2\pi)^2}. \tag{20}$$

Hence we expect the probability distribution $P(W)$ to display an edge singularity:

$$P(W) \approx \theta(W - \delta E)(W - \delta E)^{(\delta g/2\pi)^2 - 1}. \tag{21}$$

This expectation is readily confirmed by the exact solution of the problem for a local quench starting at the critical point $g = 1$. This can be obtained employing the scaling limit of the quantum Ising model in the Majorana representation

$$H_0[\varphi, \bar{\varphi}] = \int drim\varphi\bar{\varphi} - \frac{i}{2}\varphi\partial_r\varphi + \frac{i}{2}\bar{\varphi}\partial_r\bar{\varphi}, \tag{22}$$

where φ and $\bar{\varphi}$ are Majorana fermions, and the mass is related to the transverse field by $m = g - 1$. The quench consists in going from H_0 to $H_0 + V$, where $V[\varphi, \bar{\varphi}] = i\delta m\varphi(0)\bar{\varphi}(0)$.

In order to compute $\mathcal{G}$ at criticality ($m = 0$), let us use a trick provided by Zuber and Itzykson [21]. We start by computing $[\mathcal{G}(t)]^2$. Introducing two copies of the Majorana fermions, $\varphi_{1,2}$ and $\bar{\varphi}_{1,2}$, we have

$$[\mathcal{G}(t)]^2 = \langle e^{i\mathcal{H}_0 t}e^{-i(\mathcal{H}_0 + \mathcal{V})t}\rangle, \tag{23}$$

where $\mathcal{H}_0 = H_0[\varphi_1, \bar{\varphi}_1] + H_0[\varphi_2, \bar{\varphi}_2]$ and $\mathcal{V} = V[\varphi_1, \bar{\varphi}_1] + V[\varphi_2, \bar{\varphi}_2]$. The most elegant way to proceed consists now in combining the Majorana fermions into Dirac fermions $\Psi_R = (\varphi_1 + i\varphi_2)/\sqrt{2}$ and $\Psi_L = (\bar{\varphi}_1 + i\bar{\varphi}_2)/\sqrt{2}$, and then introducing a pair of nonlocal operators [22] defined as $\Psi_+(r) = [\Psi_R(r) + \Psi_L(-r)]/\sqrt{2}$, and $\Psi_-(r) = [\Psi_R(r) - \Psi_L(-r)]/\sqrt{2}i$. In these terms,

$$\mathcal{H}_0 = \int dr\Psi_+^\dagger(-i\partial_r)\Psi_+ + \Psi_-^\dagger(-i\partial_r)\Psi_-, \tag{24}$$

$$\mathcal{V} = \delta m[\Psi_+^\dagger(0)\Psi_+(0) - \Psi_-^\dagger(0)\Psi_-(0)]. \tag{25}$$

Physically, it is now evident that we have two chiral modes subject to local potential scattering of opposite sign characterized by phase shifts $\pm\delta = \pm\delta m/2$. The computation of $\mathcal{G}^2$ is now a standard problem solvable by bosonization [14,23]. We find that

$$\mathcal{G}(t) = \left[\frac{1}{1 + it}\right]^{(\delta/\pi)^2}. \tag{26}$$

The complex conjugate of this expression is readily recognized to be the characteristic function of the Gamma probability distribution

$$P(w) = \frac{w^{(\delta/\pi)^2 - 1}e^{-w}}{\Gamma[(\frac{\delta}{\pi})^2]}, \tag{27}$$

which indeed displays an edge singularity with a exponent $(\delta/\pi)^2 = [\delta g/(2\pi)]^2$ consistent with the one obtained by the cumulant expansion.

In conclusion, after elucidating the connection between the probability distribution $P(W)$ of the work done on a system in a quantum quench and the Loschmidt echo, we characterized $P(W)$ for global and local quenches of the transverse field in a quantum Ising chain. As mentioned before, the experimental measurement of $P(W)$ requires the realization of an optical absorbtion experiment in a fully controllable setting. Recent proposals for the realization of quantum spin chains using bosonic atoms in optical lattices [24] give a possible, concrete way to pursue this goal with the available experimental tools.

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[1] M. Greiner *et al.*, Nature (London) **415**, 39 (2002); M. Greiner *et al.*, Nature (London) **419**, 51 (2002).
[2] T. Kinoshita, T. Wenger, and D. S. Weiss, Nature (London) **440**, 900 (2006).
[3] M. Rigol *et al.*, Phys. Rev. Lett. **98**, 050405 (2007).
[4] W. H. Zurek, U. Dorner, and P. Zoller, Phys. Rev. Lett. **95**, 105701 (2005); A. Polkovnikov, Phys. Rev. B **72**, 161201 (R) (2005).
[5] E. Altman and A. Auerbach, Phys. Rev. Lett. **89**, 250404 (2002).
[6] P. Mazur, Physica (Utrecht) **43**, 533 (1969); E. Barouch, B.M. McCoy, and M. Dresden, Phys. Rev. A **2**, 1075 (1970).
[7] K. Sengupta, S. Powell, and S. Sachdev, Phys. Rev. A **69**, 053616 (2004).
[8] F. Iglói and H. Rieger, Phys. Rev. Lett. **85**, 3233 (2000).
[9] P. Calabrese and J. Cardy, Phys. Rev. Lett. **96**, 136801 (2006).
[10] P. Calabrese and J. Cardy, J. Stat. Mech. (2007) P10004.
[11] C. Jarzynski, Phys. Rev. Lett. **78**, 2690 (1997).
[12] J. Kurchan, arXiv:cond-mat/0007360v2.
[13] P. Talkner, E. Lutz, and P. Hänggi, Phys. Rev. E **75**, 050102 (2007).
[14] K. D. Schotte and U. Schotte, Phys. Rev. **182**, 479 (1969).
[15] R. A. Jalabert and H. M. Pastawski, Phys. Rev. Lett. **86**, 2490 (2001); Z. P. Karkuszewski, C. Jarzynski, and W. Zurek, Phys. Rev. Lett. **89**, 170405 (2002).
[16] H. T. Quan *et al.*, Phys. Rev. Lett. **96**, 140604 (2006).
[17] D. Rossini *et al.*, Phys. Rev. A **75**, 032333 (2007).
[18] A. Silva (to be published).
[19] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, 1999).
[20] A. LeClair, G. Mussardo, H. Saleur, and S. Skorik, Nucl. Phys. **B453**, 581 (1995).
[21] J. B. Zuber and C. Itzykson, Phys. Rev. D **15**, 2875 (1977).
[22] C. L. Kane, K. A. Matveev, and L. I. Glazman, Phys. Rev. B **49**, 2253 (1994).
[23] Here we set the ultraviolet cutoff $\alpha \simeq J = 1$.
[24] L. M. Duan, E. Demler, and M. D. Lukin, Phys. Rev. Lett. **91**, 090402 (2003).