Dilute One-Dimensional Spin Glasses with Power Law Decaying Interactions

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(Received 1 February 2008; revised manuscript received 6 April 2008; published 3 September 2008)

We introduce a diluted version of the one-dimensional spin-glass model with interactions decaying in probability as an inverse power of the distance. In this model, varying the power corresponds to changing the dimension in short-range models. The spin-glass phase is studied in and out of the range of validity of the mean-field approximation in order to discriminate between different theories. Since each variable interacts only with a finite number of others the cost for simulating, the model is drastically reduced with respect to the fully connected version, and larger sizes can be studied. We find both static and dynamic indications in favor of the so-called replica symmetry breaking theory.

DOI: 10.1103/PhysRevLett.101.107203 PACS numbers: 75.10.Nr, 05.70.Fh, 71.55.Jv

Mean-field spin-glass models are known to have a rather complex low-temperature phase [1], which has not been clearly observed so far in numerical simulations of finite-dimensional models with short-range (SR) interactions. Theories alternative to the mean-field (MF) one have been proposed [2], but SR systems are very tough to study analytically [3]. Numerical simulations have been, thus, extensively employed, developing more and more refined algorithms over the years, though with no conclusive indication on the nature of the spin-glass (SG) phase in finite dimensions. Long-range (LR) models are such that their lower critical dimension is lower than that of the corresponding SR model. In particular, one can have a phase transition even in one-dimensional systems, provided the range of interaction is large enough. One-dimensional spin-glass models with power-law decaying interactions actually allow us to explore both LR and SR regimes by changing the power [4–8]. These models would be perfect candidates for comparing the spin-glass phase in and out of the range of validity of the mean-field approximation. Unfortunately, since each variable interacts with all the others, numerical simulations are very computer demanding, and it is hard to get clear experimental evidence supporting a specific spin-glass theory [7,8]. We, therefore, introduce a diluted version of the model, where the mean coordination number is fixed (see also Ref. [9]). In diluting, the run time grows as the size L of the system, rather than proportionally to L^2. This is a fundamental issue because finite volume effects are strong in these models: previous studies were restricted to $L \leq 512$, while we can now thermalize considerably larger systems, thus keeping these effects under control.

We analyze the difference among the predictions on the spin-glass phase of the droplet theory [2], the trivial-nontrivial (TNT) scenario [10], and the replica symmetry breaking (RSB) theory [1]. Studying the thermodynamics, we focus on site and link overlaps, providing strong evidence that both fluctuate in the infinite volume limit. From the dynamic behavior, we learn that the four-point correlation function goes to zero at large distances when extrapolated at infinite times. In this framework, we are able to identify a characteristic lengthscale $\ell(T;t)$.

The model investigated is a one-dimensional chain of L Ising spins ($\sigma_i = \pm 1$) whose Hamiltonian reads

$$\mathcal{H} = -\sum_{i<j} J_{ij}\sigma_i\sigma_j. \tag{1}$$

The quenched random couplings J_{ij} are independent, and identically distributed random variables taking a nonzero value with a probability decaying with the distance between spins σ_i and σ_j, $r_{ij} = |i - j| \text{mod}(L/2)$, as

$$\mathbf{P}[J_{ij} \neq 0] \propto r_{ij}^{-\rho} \quad \text{for } r_{ij} \gg 1. \tag{2}$$

Nonzero couplings take value ± 1 with equal probability. We use periodic boundary conditions and a $z = 6$ average coordination number [11].

The universality class depends on the value of the exponent ρ, and it turns out to be equal to the one of the fully connected version of the model, where bonds are Gaussian distributed with zero mean and a variance depending on the distance as $\overline{J_{ij}^2} \propto r_{ij}^{-\rho}$ [4–8]. The overline denotes the average over quenched disorder.

As ρ varies, this model is known to display different statistical mechanics behaviors. For the diluted case, they are reported in Table I. In the limit $\rho \to 0$, the model is an SG on a Bethe lattice [12,13], at variance with the fully connected version where this limit is ill-defined for any $\rho < 1$. If the decay is gentle enough ($\rho \leq 4/3$), the MF approximation is exact. As it becomes steeper ($\rho > 4/3$), the MF approximation breaks down because of infrared divergences (IRD). The value $\rho = 4/3$ is shown, e.g., in Refs. [4,7], to be the threshold of validity of MF theory in the fully connected model, and it corresponds to the upper critical dimension of SR spin-glasses in absence of an external magnetic field ($D = 6$). At $\rho = 2$, the finite T

TABLE I. From infinite range to short-range behavior of the SG model defined in Eqs. (1) and (2).

$\rho < 1$	Bethe lattice like
$1 < \rho \leq 4/3$	2nd order transition, mean-field (MF)
$4/3 < \rho < 2$	2nd order transition, infrared divergence (IRD)
$\rho = 2$	Kosterlitz-Thouless or $T = 0$ phase transition
$\rho > 2$	no phase transition

transition vanishes [6], though power-law correlations might still be present [14]; this value of the exponent plays the role of the lower critical dimension in SR systems. An approximate relationship between ρ and the dimension D of SR models can be identified as follows. In LR models, the free theory in the replica space is

$$\mathcal{H} = \frac{L}{4}\int \frac{dk}{2\pi}(k^{\rho-1} + m_0^2)\sum_{a\neq b}|\tilde{Q}_{ab}(k)|^2, \tag{3}$$

where a and b are replica indices and $\tilde{Q}_{ab}(k)$ is the Fourier transform of the distance-dependent overlap matrix element $Q_{ab}(r_{ij})$. Details can be found in Ref. [7]. Comparing the critical scaling ($m_0 \propto |T - T_c| = 0$) of Eq. (3) with that of the free theory for SR spin-glass models in D dimensions ($\mathcal{H} \sim \int d^D k k^2 \mathrm{Tr} Q^2$), the following equation turns out to hold close to the upper critical dimension $\rho = 1 + 2/D$.

We simulate two replicas $\sigma_i^{(1,2)}$ using the parallel tempering algorithm [16,17]. To study the equilibrium properties, we measure site and link overlaps,

$$q = \frac{1}{L}\sum_{i=1}^{L}\sigma_i^{(1)}\sigma_i^{(2)}, \qquad q_l = \frac{1}{zL}\sum_{i,j}^{1,L} J_{ij}^2 \sigma_i^{(1)}\sigma_j^{(1)}\sigma_i^{(2)}\sigma_j^{(2)}, \tag{4}$$

and $\xi_L = [\chi_{\mathrm{sg}}/\tilde{\chi}(2\pi/L) - 1]^{1/(\rho-1)}/[2\sin(\pi/L)]$, the correlation length [18]. $\chi_{\mathrm{sg}} = L\overline{\langle q^2 \rangle}$ is the spin-glass susceptibility ($\langle\cdots\rangle$ denotes the thermal average and $\overline{\cdots}$ denotes the average over the disorder), and $\tilde{\chi}(k)$ is the Fourier transform of the four-point correlation function ($\tilde{\chi}(0) = \chi_{\mathrm{sg}}$). To compute critical properties and finite size scaling (FSS) corrections, we have used the quotient method [19]. We have computed the exponent ν from the scaling of the temperature derivative of ξ_L/L and η from the scaling of χ_{sg}. As a typical case, we show in Fig. 1 the temperature and size dependence of χ_{sg} and ξ_L. In the quotient method, the estimates of the critical exponent still depend on the lattice size: the extrapolation to infinite volume provides both their asymptotic values and the ω exponent of the leading FSS correction, $O(L^{-\omega})$. The results are summarized in Table II. The η exponent coincides with the theoretical prediction $\eta = 3 - \rho$ (η is not renormalized in the IRD regime [4,7]). Because of strong finite size effects, this check failed in previous works [8]. The ν exponent is consistent with the theoretical prediction, $\nu = 1/(\rho - 1)$, in the MF case. In the IRD regime, thermodynamic fluctuations dominate and a renormalization is necessary: at present only one-loop calculations are available [4,7], but their estimates of ν are too rough to compare with numerical data.

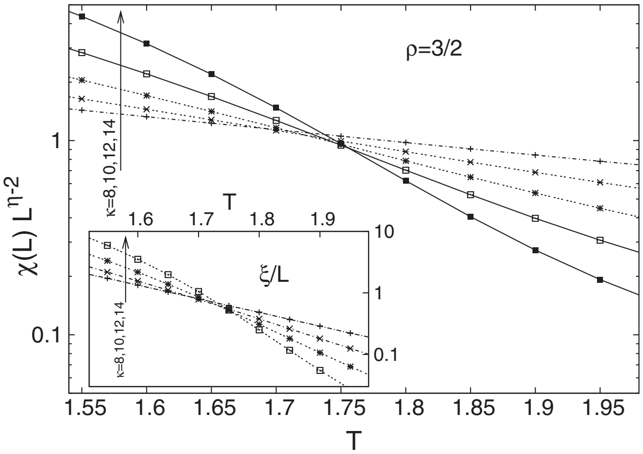

FIG. 1. $\rho = 3/2$, IRD regime. Plot of $L^{\eta-2}\chi_{\mathrm{sg}}$ vs T. Inset: ξ_L/L vs T. Sizes are $L = 2^\kappa$, with $\kappa = 8, 10, 12, 14$.

In the spin-glass phase ($T < T_c$), site and link overlap distributions, $P(q)$ and $P_l(q_l)$, can be used as hallmarks to discriminate among different theories for finite-dimensional spin glasses. Indeed, three cases are contemplated in the literature. 1. Droplet theory: one state; both distributions are delta-shaped. 2. TNT scenario: many states (q fluctuates), but dropletlike excitations (q_l fluctuations vanish for large sizes); $P(q)$ is broad and $P_l(q_l)$ is delta-shaped. 3. RSB theory: many states with space-filling excitations; both distributions are broad.

Distributions $P(q)$ and $P_l(q_l)$ for $T \simeq 0.4T_c$ are plotted in Figs. 2 and 3 in a case where MF is exact ($\rho = 5/4$) and in an IRD case ($\rho = 3/2$), respectively. In both cases, we see two peaks in the $P_l(q_l)$ for large sizes. Out of MF, such a result would have been impossible to observe in this model with sizes smaller than $L = 2^{12}$.

Both distributions seem to be broad, but their thermodynamic limits must be taken carefully. While it is easy to prove that $P(q)$ is not bimodal as $L \to \infty$ [lower insets in Figs. 2 and 3 show that $P(0)$ becomes size independent], the limit of $P_l(q_l)$ is more difficult to extract from finite size data, since its variance converges to a small value, see upper insets of Figs. 2 and 3 [20]. We provide, thus, an alternative method of analysis, testing the hypothesis that both q and q_l are equivalent measures of the distance among states [21]. The simplest relation is $q_l = q_{\mathrm{aux}} \equiv A + Bq^2 + C\sqrt{1 - q^2}z$, where z is a normal random vari-

TABLE II. Estimates of critical temperature and exponents.

	ρ	"D"	T_c	$1/\nu$	η	η (th.)	ω
MF	5/4	8	2.191(5)	0.28(2)	1.751(8)	1.75	0.40(2)
IRD	3/2	4	1.758(4)	0.25(3)	1.502(8)	1.5	0.60(6)
IRD	5/3	3	1.36(1)	0.19(3)	1.32(1)	$1.3\bar{3}$	0.8(1)

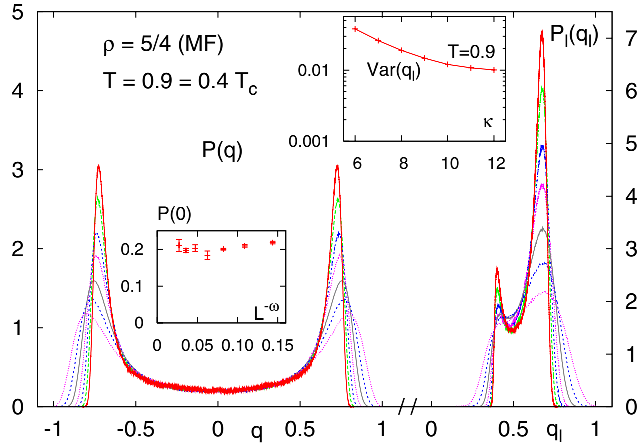

FIG. 2 (color online). $P(q)$ and $P_l(q_l)$ at $\rho = 5/4$ (MF). $T = 0.9 \simeq 0.4T_c$ and $L = 2^\kappa$ with $\kappa = 7, \ldots, 13$. Upper inset: variance of $P(q_l)$ vs size (double-log scale). Lower inset: $P(0)$ vs $L^{-\omega}$.

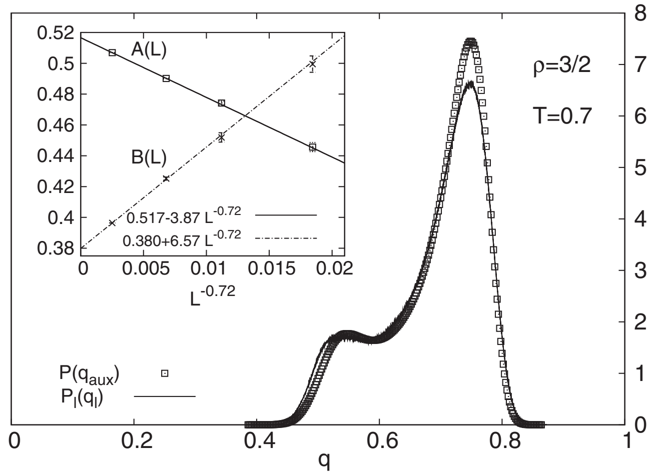

FIG. 4. Distributions of q_l (line) and $q_{\rm aux}$ (empty squares) for $\rho = 3/2$, $L = 2^{12}$ and $T = 0.7$. Inset: A and B vs $L^{-0.72}$ obtained by measuring the Kullback-Leibler divergence between the two distributions for $L = 2^\kappa$, $\kappa = 6, 8, 10, 12$.

able mimicking finite size effects, and A, B and C are fitting parameters. Such a relation is satisfied in the Sherrington-Kirkpatrick model, with $A = C = 0$, and it is a good approximation for the SR spin glass in $D = 3$ [22,23]. For each value of L, at $\rho = 3/2$ and $T = 0.7$, we compute the best fitting parameters by minimizing the symmetrized Kullback-Leibler divergence (KLD) [24] between the distribution of q_l and that of $q_{\rm aux}$. In Fig. 4, we compare optimal distributions for $L = 2^{12}$, which should coincide if the relation $q_l = q_{\rm aux}$ held. This provides strong evidence for a nontrivial link overlap distribution as long as the B parameter converges to a non zero value for $L \longrightarrow \infty$, as one can verify in the inset of Fig. 4 where we plot A and B vs an inverse power of L (C and the optimal KLD go to zero, as expected).

As a complementary approach, we look at the off-equilibrium four-point correlation function

$$C_q(x, t) = \frac{1}{L} \sum_{i=1}^{L} \overline{\langle \sigma_i^{(1)}(t)\sigma_i^{(2)}(t)\sigma_{i+x}^{(1)}(t)\sigma_{i+x}^{(2)}(t)\rangle}. \quad (5)$$

For very large distances, the fastest decay expected goes like $x^{-\rho}$, because of LR interactions. For intermediate distances, up to a length $\ell(t)$, we observe a slower decay $x^{-\alpha}$, with $0 < \alpha < \rho$, which is incompatible with the onset of a plateau at $q_{\rm EA}^2$ in the large times limit. This suggests to use the function

$$Ax^{-\alpha}[1 + (x/\ell)^{\delta(\rho-\alpha)}]^{-1/\delta} \quad (6)$$

to interpolate $C_q(x, t)$ data at a fixed time t. The fits are very good, and their quality can be appreciated in Fig. 5 for an IRD ($\rho = 3/2$) system of size $L = 2^{17}$. The crossover length ℓ plays a role similar to the correlation (or coherence) length in short-range spin glasses [25]. We allow the

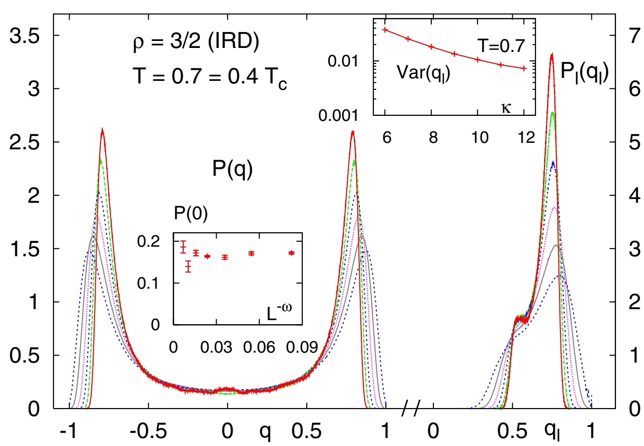

FIG. 3 (color online). $P(q)$ and $P_l(q_l)$ at $\rho = 3/2$ (IRD). $T = 0.7 \simeq 0.4T_c$ and $L = 2^\kappa$ with $\kappa = 7, \ldots, 12$. Upper inset: variance of $P(q_l)$ vs size (double-log scale). Lower inset: $P(0)$ vs $L^{-\omega}$.

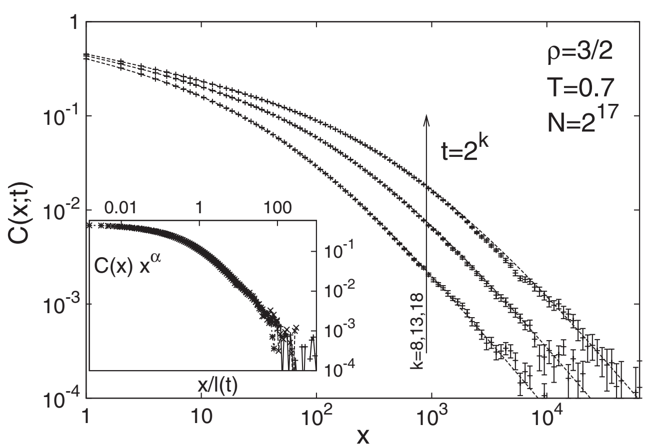

FIG. 5. $C_q(x, t)$ for $\rho = 3/2$, $L = 2^{17}$ at $T = 0.7 \simeq 0.4T_c$ and different times $t = 2^8$, 2^{13}, 2^{18}. The curves are fits to Eq. (6). Inset: data collapse by plotting $C(x; t)x^\alpha$ vs $x/\ell(t)$, for all times between 2^8 and 2^{18}.

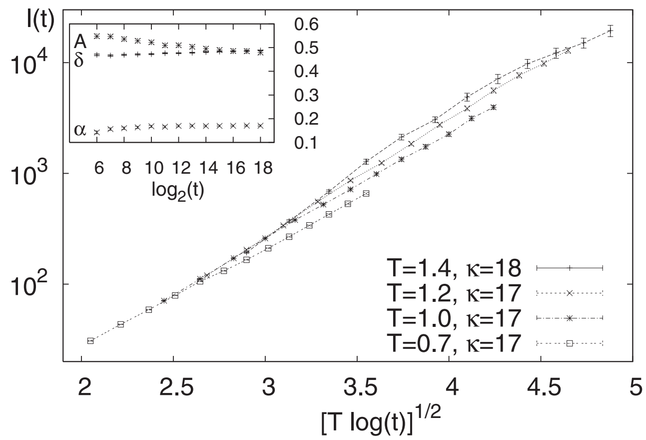

FIG. 6. Crossover length $\ell(t)$ vs $\sqrt{T \log t}$ for $\rho = 3/2$, $L = 2^\kappa$, and T between $0.7 \simeq 0.4T_c$ and $1.4 \simeq 0.79T_c$. Inset: parameters A, α, and δ vs time at $T = 0.7$.

fitting parameters A, α, and δ to depend on time. Nonetheless, we observe (see inset of Fig. 6) that they become stationary for large times: this is strong evidence that Eq. (6) has significant and robust behavior. The growth of $\ell(t)$ with time for some temperatures below T_c is plotted in Fig. 6. The length $\ell(t)$ reaches very large values ($>10^4$) with respect to previous studies on spin-glass models [25]. In this region, $\ell(t)$ is very well fitted by the phenomenological law $a(T)\exp[b(T)\sqrt{T \log t}]$, with a and b not very dependent on the temperature; this seems reasonable since in activated processes, the typical scaling variable is $T\log(t)$. We also tried to fit the previous $\ell(t)$ data with a law for the characteristic time necessary to nucleate a droplet-like excitation of size ℓ: $\tau(\ell) = A(T)\ell^{z_c} \times \exp[\Upsilon(T)\ell^{\psi}]$, where the power-law factor dominates near the transition $[\lim_{T\to T_c}\Upsilon(T) = 0]$ and the exponential term governs the low-temperature regime. The critical exponents z_c and ψ are predicted not to depend on T. The data shown in Fig. 6 are not compatible with this scaling law for any temperature-dependent $A(T)$ and $\Upsilon(T)$.

In conclusion, we have introduced a model which is easy to simulate and allows to probe the spin-glass phase beyond MF. In this regime, we observe that both site and link overlaps fluctuate for large sizes. In the large times limit, the out-of-equilibrium four-point function $C_q(x, t)$ tends to a well defined function that displays a power-law decay to zero and is incompatible with the onset of a plateau at any large x. These observations are consistent with the clustering properties of the RSB theory. The bond diluteness of the model under investigation strongly reduces simulation times and allows to thermalize systems of sizes large enough to clearly discern the double peak structure of $P_l(q_l)$. Both droplet and TNT proposal appear not consistent with a FSS analysis over large sizes and with the behavior of the four-point correlation function and the related coherence length.

We thank Silvio Franz for useful discussions and suggestions. This work has been partially supported by MEC, Contracts Nos. FIS2006-08533-C03 and FIS2007-60977. Part of simulations were performed in the BIFI cluster.

[1] G. Parisi, Phys. Lett. A **73**, 203 (1979); Phys. Rev. Lett. **50**, 1946 (1983).
[2] D. S. Fisher and D. A. Huse, Phys. Rev. Lett. **56**, 1601 (1986).
[3] T. Temesvari *et al.*, Eur. Phys. J. B **11**, 629 (1999).
[4] G. Kotliar *et al.*, Phys. Rev. B **27**, 602 (1983).
[5] A. C. D. van Enter and J. L. van Hemmen, J. Stat. Phys. **32**, 141 (1983); **39**, 1 (1985); M. A. Moore, J. Phys. A **19**, L211 (1986); R. N. Bhatt and A. P. Young, J. Magn. Magn. Mater. **54–57**, 191 (1986).
[6] M. Campanino *et al.*, Commun. Math. Phys. **108**, 241 (1987).
[7] L. Leuzzi, J. Phys. A **32**, 1417 (1999).
[8] H. G. Katzgraber and A. P. Young, Phys. Rev. B **67**, 134410 (2003); **68**, 224408 (2003).
[9] S. Franz and G. Parisi, Europhys. Lett. **75**, 385 (2006).
[10] F. Krzakala and O. C. Martin, Phys. Rev. Lett. **85**, 3013 (2000).
[11] Links are generated by repeating $zL/2$ times the following process: choose randomly 2 spins at distance r with probability $r^{-\rho}/\sum_{i=1}^{L/2} i^{-\rho}$; if they are already connected, repeat the process, otherwise connect them.
[12] L. Viana and A. J. Bray, J. Phys. C **18**, 3037 (1985).
[13] M. Mézard and G. Parisi, Eur. Phys. J. B **20**, 217 (2001).
[14] The fully-connected model with $\rho = 2$ and no quenched disorder displays a Kosterlitz-Thouless transition [15].
[15] E. Luijten and H. Meßingfeld, Phys. Rev. Lett. **86**, 5305 (2001).
[16] K. Hukushima and K. Nemoto, J. Phys. Soc. Jpn. **65**, 1604 (1996).
[17] For the critical behavior, we simulated sizes up to 2^{14} with about 20 temperatures in the intervals [1.65, 2.65] ($\rho = 5/4$), [1.4, 2.0] ($\rho = 3/2$), and [0.7, 1.7] ($\rho = 5/3$); number of disorder samples is $O(3 \times 10^5)$ for $L = 2^6$ and $O(2 \times 10^4)$ for $L = 2^{14}$. To study low-temperature properties, we further simulated about 40 temperatures with $T \in [0.8, 2.65]$ and $L \leq 2^{13}$ ($\rho = 5/4$), and $T \in [0.7, 2.7]$ and $L \leq 2^{12}$ ($\rho = 3/2$); number of samples at the largest sizes is $O(10^3)$; thermalization times range from 2^{10} ($L = 2^6$) to 2^{21} ($L = 2^{13}$) for $\rho = 5/4$, and from 2^{13} ($L = 2^6$) to 2^{20} ($L = 2^{12}$) for $\rho = 3/2$.
[18] F. Cooper *et al.*, Nucl. Phys. B **210**, 210 (1982).
[19] H. G. Ballesteros *et al.*, Phys. Rev. B **58**, 2740 (1998).
[20] Data for $\mathrm{Var}(q_l)$ are not compatible with a simple power law ($\chi^2 = 40.9$ for 5 d.o.f. at $\rho = 3/2$, Fig. 3). The large size limit is finite, equal to 0.0021(4) for $\rho = 3/2$.
[21] G. G. Athanasiu *et al.*, Phys. Rev. B **35**, 1965 (1987).
[22] G. Hed and E. Domany, Phys. Rev. B **76**, 132408 (2007).
[23] P. Contucci *et al.*, Phys. Rev. Lett. **99**, 057206 (2007).
[24] S. Kullback and R. A. Leibler, Ann. Math. Stat. **22**, 79 (1951).
[25] J. Kisker *et al.*, Phys. Rev. B **53**, 6418 (1996); E. Marinari *et al.*, J. Phys. A **33**, 2373 (2000); L. Berthier and J.-P. Bouchaud, Phys. Rev. B **66**, 054404 (2002).