Large-N_c Confinement and Turbulence

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We suggest that the transition that occurs at large N_c in the eigenvalue distribution of a Wilson loop may have a turbulent origin. We arrived at this conclusion by studying the complex-valued inviscid Burgers-Hopf equation that corresponds to the Makeenko-Migdal loop equation, and we demonstrate the appearance of a shock in the spectral flow of the Wilson loop eigenvalues. This picture supplements that of the Durhuus-Olesen transition with a particular realization of disorder. The critical behavior at the formation of the shock allows us to infer exponents that have been measured recently in lattice simulations by Narayanan and Neuberger in $d = 2$ and $d = 3$. Our analysis leads us to speculate that the universal behavior observed in these lattice simulations might be a generic feature of confinement, also in $d = 4$ Yang-Mills theory.

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Many efforts continue to be devoted to the study of QCD in the limit of a large number of colors, after the initial suggestion by 't Hooft [1]. This, in part, is due to the general belief that the large N_c limit captures the essence of confinement, one of the most elusive of QCD properties. At the same time the theory simplifies considerably in the large N_c limit: fluctuations die out and the measure of the partition function becomes localized at one particular configuration, making the large N_c limit akin to a classical approximation [2].

This Letter will discuss the large N_c limit of Yang-Mills theory in 2 dimensions, but we have good reasons to believe that much of our analysis can be extended to higher dimensions. There are many equivalent approaches to multicolor Yang-Mills theory in $d = 2$. For definiteness, we shall refer to the known [3] formulation in terms of free random variables [4,5]. This translates $d = 2$ Yang-Mills theory onto the large N_c matrix model, where the size of the unitary matrix is identified with the number of colors. More specifically, the basic observable that we shall consider is the Wilson loop along a (simple) curve C

$$W(A) = \left\langle P \exp\left(i \oint_C A_\mu dx_\mu \right) \right\rangle, \tag{1}$$

where the averaging is over the Yang-Mills measure, and we have made explicit that W depends in fact only on the area A enclosed by C, to within a normalization [6]. The matrix W is unitary, with eigenvalues of the form $\lambda = \exp(i\theta)$ that are distributed, in the limit $N_c \to \infty$, according to an average density $\rho(\theta, A)$.

Recently, Narayanan and Neuberger [7] studied the behavior of $W(A)$ as a function of A in the large N_c limit. They observed that for small loops (which probe short distance, perturbative physics), the spectrum does not cover the whole unit circle, but exhibits a gap; in contrast, for very large loops (which probe long distance, nonperturbative physics) the spectrum covers uniformly the unit circle (gapless phase). Since the crossover region is becoming infinitely thin [7] in the limit $N_c \to \infty$, one is tempted to try and explain the transition using classical concepts only, which is precisely what we aim at in this Letter. In fact this behavior of the spectrum agrees with the order (gapped)-disorder(gapless) transition, proposed long ago in the context of large N_c two-dimensional Yang-Mills theory by Durhuus and Olesen [6] and based on the explicit solutions of corresponding Makeenko-Migdal equations [8]. Surprisingly, a similar critical behavior has been observed also in $d = 3$ dimensions and conjectured to hold in $d = 4$ large N_c Yang-Mills theory [7]. In this Letter we suggest the general mechanism for such a transition by tracing the complex singularities of the eigenvalue flow equation and demonstrating the appearance of "spectral shocks" that may signal the transition to a turbulent state.

The spectral density is not available in analytic form, but the moments

$$w_n(A) \equiv \langle \mathrm{tr} W(A)^n \rangle = \int_{-\pi}^{+\pi} d\theta e^{in\theta} \rho(\theta, A) \tag{2}$$

are. An explicit, compact form for these moments is given in Ref. [3] in terms of an integral representation

$$\begin{aligned} w_n(A) &= \frac{1}{n} \oint \frac{dz}{2\pi i} (1 + 1/z)^n \exp(-nA(z + 1/2)) \\ &= \frac{1}{n} L^1_{(n-1)}(nA) \exp(-nA/2), \end{aligned} \tag{3}$$

where the representation of Laguerre polynomials, used in the second line, allows connection to results known already 25 years ago [6,9]. The Durhuus-Olesen transition can be seen by studying the asymptotic behavior of these Laguerre polynomials, using a saddle point analysis of their integral representation [10,11]. The result is surprising: for a loop area below the critical value $A_c = 4$, the moments oscillate and decay like $n^{-3/2}$, while for $A > A_c$ the moments decay exponentially with n, modulo similar power behavior. Both regimes are separated by double scaling limit. There exists vast literature on this subject [9]. Here we would only like to stress that the transition is subtle, and popular arguments that two-dimensional confinement is trivial, and perturbative (linear potential), can easily lead to paradoxes when, e.g., instanton effects are not taken into account [9].

In order to analyze the nature of the transition, we consider the following function [3]:

$$F(\theta, A) = i(e^{i\theta} G(e^{i\theta}, A) - \tfrac{1}{2}), \tag{4}$$

where $G(z)$ is the resolvent

$$G(z, A) = \int_{-\pi}^{+\pi} d\alpha \frac{\rho(\alpha, A)}{z - e^{i\alpha}}. \tag{5}$$

The function $F(\theta, A)$ is analytic in the complex θ plane, with a discontinuity across the real axis proportional to the spectral density, $\rho(\theta, A) = (1/\pi)\,\mathrm{Im}F(\theta - i0^+, A)$. A simple calculation yields the explicit expressions [6]

$$\begin{aligned} F(\theta, A) &= i\left(\frac{1}{2} + \sum_{n=1}^{+\infty} w_n(A) e^{-in\theta}\right) \\ &= \frac{1}{2}\int_{-\pi}^{+\pi} d\alpha \rho(\alpha, A) \cot\left(\frac{\theta - \alpha}{2}\right), \end{aligned} \tag{6}$$

whose imaginary part gives the spectral density in the form

$$\rho(\theta, A) = \frac{1}{2\pi}\left(1 + \sum_{n=1}^{+\infty} 2 w_n(A) \cos(n\theta)\right), \tag{7}$$

in agreement with Eq. (2) above (we have used the fact that the moments are real and that $\rho(-\theta) = \rho(\theta)$).

It can be shown that the function F obeys the following equation [3,6,9]

$$\partial_A F + F \partial_\theta F = 0. \tag{8}$$

This is the so-called complex Burgers equation in the inviscid limit. This equation is analogous to the real Burgers equation of fluid dynamics (with A playing the role of time, θ that of a coordinate, and F of a velocity field). The complex Hopf-Burgers equation is omnipresent in free random variables calculus [4]. This, and similar integro-differential equations also appear frequently as one-dimensional models for quasigeostrophic equations, describing, e.g., the dynamics of the mixture of cold and hot air and the fronts between them. Another reason why the complex Burgers equation is much studied is the fact that the structure of quasigeostrophic equations resembles 3D Euler equations, if one substitutes the velocity by the vorticity [12]. Here, we shall take advantage of the abundant mathematical studies of the complex Burgers equation to suggest a connection between signals of turbulence (commonly associated with the blowup of the solution in finite time) and the problem of spectral flow of the eigenvalues of Wilson loop operators. We shall, in particular, adapt the proof of the blowup from Ref. [13], using the method of complex characteristics, and tracing singularities in the complex plane. The observation of shock formation in the complex Burgers and similar equations was confirmed by other methods; see, e.g., [14].

In the present case, the method of characteristics provides the following implicit solution in terms of an auxiliary function $\xi(A, \theta)$:

$$F(A, \theta) = F_0(\xi(A, \theta)), \qquad \theta = \xi + A F_0(\xi), \tag{9}$$

where F_0 is given by the initial condition

$$F_0(\theta) = F(A = 0, \theta). \tag{10}$$

This implicit solution is well defined as long as the mapping between θ and ξ remains regular. However, singular points occur when $d\theta/d\xi = 0$, that is, for ξ_c solution of

$$1 + A F_0'(\xi_c) = 0. \tag{11}$$

A lot of information on the solution, and, in particular, the occurrence of a blowup, can be inferred from the study of the location of these singular points.

We begin our analysis with the ordered (gapped) state, and recover known results (see, e.g., the second Ref. [10]) in an easy and direct way. We start from an initial condition peaked at eigenvalue $\lambda = 1$, i.e., $\rho(\theta, 0) = \delta(\theta)$ [corresponding to $A = 0$ in Eq. (3)]. Then, from (6), $F_0(\theta) = 1/2 \cot\theta/2$, and from (11) $\sin^2 \xi_c/2 = A/4$. In the vicinity of the singularity the characteristics behave as

$$\theta = \theta_c + (\xi - \xi_c)^2 \sqrt{1/A - 1/4}, \tag{12}$$

where $\theta_c = \xi_c + \sqrt{A(1 - A/4)}$ and $\xi_c = 2\arcsin(\sqrt{A}/2)$. The spectral density is easily deduced from the imaginary part of F or equivalently F_0 [see (10)]. A simple analysis then reveals that $\rho(\theta, A) = 0$ when $\theta > \theta_c$, while for $\theta \lesssim \theta_c$, $\rho(\theta, A) \sim \sqrt{\theta_c - \theta}$. In other words, θ_c determines the edge of the spectrum which, as long as $A < 4$, presents a gap.

At the closure of the gap, i.e., for $A = A_c = 4$, the second derivative in the expansion (12) vanishes, and we have instead

$$\theta = \pi - \frac{1}{3A_c}(\xi - \xi_c)^3, \tag{13}$$

with the spectral density $\rho(\theta, A_c) \sim (\pi - \theta)^3$ for $\theta \lesssim \pi$.

From this point on, as A keeps increasing, $\mathrm{Re}\,\xi_c$ remains equal to π, but the initial singularity splits into two complex conjugate ones that move away to $\pm i\infty$, leading eventually to a uniform spectral density at large A. This behavior is reminiscent of the turbulent inverse spectral cascade (alike in two-dimensional turbulence [15]) which, as A grows, *suppresses* higher Fourier modes, leaving in the $A = \infty$ limit only the longest wavelength mode corresponding here to the constant density $\rho(\theta) = 1/2\pi$.

To demonstrate the inverse spectral cascade, we consider, following [13], a small perturbation of the form

$$\rho(\theta, A_0) = \frac{1}{2\pi}(1 + 2\epsilon \cos\theta) + \mathcal{O}(\epsilon^2), \qquad (14)$$

where $A_0 \gg 1$. This is of the form (7) with $w_1 = \epsilon$, and all other moments vanishing exponentially. It follows from (6) that $F_0(\xi) = \frac{i}{2}[1 + 2\epsilon \exp(-i\xi)]$, $\theta = \xi + (A - A_0)F_0(\xi)$, and a singularity occurs when $\exp(i\xi_c) = -\epsilon(A - A_0)$. In the vicinity of the singularity

$$\theta = \theta_c + \frac{i}{2}(\xi - \xi_c)^2, \qquad (15)$$

where $\theta_c = \xi_c + (A - A_0)F_0(\xi_c)$. We have two solutions depending on whether $A > A_0$ or $A < A_0$. In the first case

$$\theta_c = \pi - i\left(1 - \frac{A - A_0}{2} + \ln\epsilon(A - A_0)\right), \qquad (16)$$

whereas in the second case

$$\theta_c = -i\left(1 + \frac{A_0 - A}{2} + \ln\epsilon(A_0 - A)\right). \qquad (17)$$

In the first case, the singularity is initially (when $A = A_0$) at $\theta_c = \pi + i\infty$, and for small ϵ ($\epsilon < 1/2$), it remains complex, and returns to $+i\infty$ as $A \to \infty$: asymptotically the effect of the perturbation vanishes, and only the constant mode survives. However in the second case, the initial singularity at $+i\infty$ moves towards the real axis as A decreases, and reaches it in a finite time A^* given by

$$0 = 1 + \frac{A_0 - A^*}{2} + \ln\epsilon(A_0 - A^*). \qquad (18)$$

At this point we have a blowup [13] of the type already encountered earlier at the closure of the gap. In fact, this phenomenon is quite generically associated to the motion (as A decreases) of complex conjugate singularities (square root branch points) towards the real axis, that eventually merge on the real axis (into a third order branch point).

This generic phenomenon has an exact analog in optics [16], where light rays play the role of the characteristics. The singularities lines are there the caustics, and the merging of singularities discussed above corresponds to the merging of "twofold" caustics into a "cusp" (in the terminology of catastrophe theory). The essence of the cusp singularity is captured by Pearcey's function

$$P(\xi, \eta) = \int_{-\infty}^{\infty} dt \exp(i(t^4/4 + \xi t^2/2 + \eta t)), \qquad (19)$$

whereas the square-root-type singularity is described by the well-known Airy function. This analogy can be used in order to explain the origin of the universal function for Yang-Lee zeroes for $d \geq 2$ dimensional Yang-Mills theory obtained in Ref. [7]. The explicit comparison will be presented elsewhere [17].

The location of the singularities in the complex plane also determines the asymptotic behavior of the moments, according to general arguments [18]. For complex θ_c, one may write $\theta_c \equiv \theta^* + i\Delta(A)$, with θ^* real; the singularity is of the form $(\theta - \theta_c)^\mu$, where μ is not an integer (branch point singularity). It follows that the large n Fourier coefficients, i.e., the moments w_n, behave as

$$w_n = |n|^{-(\mu+1)} e^{-n\Delta(A)} \mathrm{Re} e^{in\theta^*}. \qquad (20)$$

Thus, the position of the singularity determines the width of an analytical strip, controlled by the value of $\Delta(A)$. In the gapped phase, the singularity is of the square root type, but always on the real axis. Then, since $\Delta = 0$, the moments only oscillate, accompanied by power law $n^{-3/2}$. In the gapless phase, the same power law is accompanied by exponential damping, since in this case the singularity is complex [with a nonvanishing $\Delta(A)$]. At the critical point, a generic cubic singularity [see Eq. (13) and [19,20]] appears in a narrow interval where $1 \ll n \ll 1/\Delta(A)$ (just approaching the singularity from above) and the scaling goes like

$$w_n \sim |n|^{-4/3}. \qquad (21)$$

This generic picture agrees with detailed calculations using the analytic form of the moments given above, based on asymptotics of Laguerre polynomials [7,10,11]. We would like to stress, however, that the method of tracing complex singularities is general and therefore we expect it to hold also in higher dimensions. The "speed" with which the singularities move may depend on the dimensionality, and possibly on other features of the loops, but for large loops it is natural to expect that the dominant control parameter will remain the area of the loop. We may therefore speculate that, also in higher dimensions, the disordered phase may be caused by an inverse spectral cascade.

One may also argue that the universal nature of the critical behavior, conjectured and observed in Ref. [7], may find its origin in the fact that since the critical behavior happens in a very narrow analyticity strip, one can expect universal (ergodic) behavior, alike in several models of disorder in mesoscopic physics. If this is the case, one may expect that simple schematic models (e.g., matrix models) may define this class of universality. This seems to be indeed the case. A particular illustration is provided by the matrix model proposed by Janik and Wieczorek [20], hereafter the JW model. The model stems from the general construction of multiplicative free evolution [21], where increments are mutually free in the sense of Voiculescu. The unitary realization in the JW model corresponds to matrix value unitary random walk, where the evolution operator is the ordered string of consecutive multiplications of infinitely large unitary matrices

$$W = \left\langle \prod_{i}^{K} U_i \right\rangle, \tag{22}$$

where $U_i = \exp i\sqrt{t/K}H_i$, with H_i a Hermitian random matrix. The model is a random matrix generalization of the multiplicative random walk performed in K steps during "time" t. In the continuum limit $K \to \infty$, the model is exactly solvable. The solution coincides exactly with the two-dimensional QCD, provided one identifies t with the area of the Wilson loop, modulo a normalization [22].

Recently, Neuberger and Narayanan [7] have observed that large N_c Yang-Mills lattice simulations in $d = 2$ and $d = 3$ demonstrate the same critical scaling at the closure of the gap as in the JW model and have conjectured that this model establishes a universality class for $d = 4$ large N_c Yang-Mills theory as well. That the whole dynamics of complicated nonperturbative QCD can be reduced *in some spectral regime* to matrix model is not new—a notable case is the universal scaling of spectral density of Euclidean Dirac operator for sufficiently small eigenvalues, where the spectrum belongs to a broad universality class of corresponding chiral models [23]. In fact the present analysis leads us to expect that in the very narrow spectral window around $\lambda = -1$ a universal oscillatory regime precedes the formation of the spectral shock, in qualitative analogy to similar spectral oscillations of the quark condensate before the spontaneous breakdown of chiral symmetry, based on the Banks-Casher relation [24].

In this Letter, we have proposed to view confinement-deconfinement transition in multicolor Yang-Mills theory as an order-disorder phenomenon, where the transition to disorder is caused by (inverse) turbulence in the spectral flow of Wilson loop operators. This picture corroborates the picture of order-disorder transition that Durhuus and Olesen envisioned many years ago, and supplements it with a detailed model for building the disorder, based on the development of the inverse spectral cascade. It will be interesting to incorporate into this picture the effects of finite N_c and matter fields. We expect that finite N_c effects will contribute to the appearance of an effective spectral viscosity ν_s, which will smoothen the shocks, but will not destroy them [17].

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