Absence of Complete Finite-Larmor-Radius Stabilization in Extended MHD

P. Zhu, D. D. Schnack, F. Ebrahimi, E. G. Zweibel, M. Suzuki, C. C. Hegna, and C. R. Sovinec
Center for Plasma Theory and Computation, University of Wisconsin-Madison, Madison, Wisconsin 53706, USA
(Received 29 February 2008; published 22 August 2008)

The dominant finite-Larmour-radius (FLR) stabilization effects on interchange instability can be retained by taking into account the ion gyroviscosity or the generalized Ohm's law in an extended MHD model. However, recent simulations and theoretical calculations indicate that complete FLR stabilization of the interchange mode may not be attainable by ion gyroviscosity or the two-fluid effect alone in the framework of extended MHD. For a class of plasma equilibria in certain finite-β or nonisentropic regimes, the critical wave number for complete FLR stabilization tends toward infinity.

DOI: 10.1103/PhysRevLett.101.085005 PACS numbers: 52.30.Cv, 52.30.Ex, 52.35.Py, 52.65.Kj

It is well known that the kinetic effects due to the finite Larmor radius (FLR) are able to stabilize the interchange mode in a weakly unstable plasma under gravity [1–4]. The dominant FLR stabilization effects on the gravitational instability (also known as g mode, or the magnetic Rayleigh-Taylor instability) can be retained by taking into account the ion gyroviscosity or the generalized Ohm's law in an extended MHD model [5,6]. Recently Ferraro and Jardin [7] extended earlier work of Roberts and Taylor [5] by including effects of plasma compression. They found the FLR effects due to ion gyroviscosity alone can completely stabilize the g mode of the isothermal equilibrium they considered in all plasma β regimes.

The extended MHD model has been widely applied in simulation studies of edge localized modes (ELMs) in tokamaks. The FLR stabilization of interchangelike, high-n ballooning modes in extended MHD provides a natural cutoff of the high-n spectrum without resorting to numerical or artificial dissipation for ELM simulations. Direct benchmarking between theory and extended MHD codes for the linear ballooning instabilities in ELMs has not been conclusive due to the complexity of the edge tokamak plasma equilibrium involved [8]. On the other hand, the FLR stabilization of g mode may provide a simpler case for benchmarking between theory and codes, while serving as a paradigm for FLR stabilization in more complicated situations, as suggested by Schnack *et al.* [9].

In a recent code verification effort, Schnack and Kruger [10] computed the linear growth of a g mode using the NIMROD code with the implementation of the extended MHD model [11]. For the particular equilibrium considered, however, they did not find the complete FLR stabilization predicted by the earlier theories that were based on extended MHD [5–7]. In order to resolve the discrepancy, we revisited the analytical dispersion relation of the pure interchange g mode, in the model of compressible extended MHD, for a general shearless slab configuration. In this Letter, we provide a calculation to clarify the prior simulation results.

Consider the following extended MHD model in a Cartesian coordinate system as in [5]:

$$\frac{d\rho}{dt} = -\rho\nabla\cdot\mathbf{u} \tag{1}$$

$$\frac{d\mathbf{u}}{dt} = -\nabla p + \mathbf{J}\times\mathbf{B} + \rho\mathbf{g} - \nabla\cdot\boldsymbol{\pi}_i \tag{2}$$

$$\frac{dp}{dt} = -\gamma p\nabla\cdot\mathbf{u} \tag{3}$$

$$\frac{\partial\mathbf{B}}{\partial t} = -\nabla\times\mathbf{E} \tag{4}$$

$$\mu_0\mathbf{J} = \nabla\times\mathbf{B} \tag{5}$$

$$\mathbf{E} = -\mathbf{u}\times\mathbf{B} + \frac{\lambda}{ne}(\mathbf{J}\times\mathbf{B} - \nabla p_e) \tag{6}$$

$$(\pi_i)_{xx} = -(\pi_i)_{yy} = -\frac{\delta p_i}{2\Omega}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) \tag{7}$$

$$(\pi_i)_{xy} = (\pi_i)_{yx} = \frac{\delta p_i}{2\Omega}\left(\frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y}\right), \tag{8}$$

where $d/dt = \partial/\partial t + \mathbf{u}\cdot\nabla$, γ is the adiabatic index, $\mathbf{g}$ the gravity, n the number density, p_i (p_e) the ion (electron) pressure, p the total pressure ($p = p_i + p_e = \tau p + p_e$, with $\tau = p_i/p$), Ω the ion gyrofrequency ($\Omega = eB/m_i$, with m_i being ion mass), and the rest of the symbols are conventional. We also use two multipliers λ and δ to track the two-fluid and the ion gyroviscosity effects, respectively.

A two-fluid static equilibrium is specified as follows:

$$\mathbf{u} = 0 \tag{9}$$

$$\mathbf{B} = B\mathbf{e}_z \tag{10}$$

$$\frac{d}{dx}\left(p + \frac{B^2}{2}\right) = \rho\mathbf{g}\cdot\mathbf{e}_x = \rho g \tag{11}$$

$$ne\mathbf{E} = \nabla p_i - \rho\mathbf{g}. \tag{12}$$

The equilibrium is assumed to vary only in x, and $\mathbf{e}_i (i = x, y, z)$ is the basis vector in each Cartesian direction. The pure interchange perturbation for the g mode has the form

$$\mathbf{u} = [u_x(x)\mathbf{e}_x + u_y(x)\mathbf{e}_y]e^{ik_y y - i\omega t} \tag{13}$$

and it satisfies the local approximation ordering: $k_y L_A \sim \epsilon$, $k_y d_i \sim \lambda \sim \delta$, $u_y \sim \epsilon u_x$, $\epsilon \ll 1$, where $L_A = |d\ln A/dx|^{-1}$ is the spatial scale of field A in x direction, and $d_i = \sqrt{p_i/\rho}/\Omega$ is the ion Larmor radius.

When both the ion gyroviscosity tensor and the generalized Ohm's law are kept in the extended MHD model, to the lowest order in ϵ we obtain the local dispersion relation for the pure interchange g mode

$$\omega(\omega^2 + \omega_{*\mathrm{FLR}}\omega + \Gamma^2_{\mathrm{FLR}}) + D_{\mathrm{FLR}} = 0, \tag{14}$$

where

$$\omega_{*\mathrm{FLR}} = \frac{k_y}{\Omega}\frac{\delta\tau[(1+\gamma\beta)(1+\beta)\frac{p'}{\rho} - (2+\gamma\beta)g\beta] - \lambda[g - \tau\frac{p}{\rho}(\ln\frac{p}{\rho^\gamma})' + \frac{k_y^2\delta^2\tau^2}{4\Omega^2}\frac{p^2}{\rho^2}\frac{\rho'}{\rho}]}{(1+\gamma\beta)(1+\frac{k_y^2\delta^2\tau^2}{4\Omega^2}\frac{p}{\rho}\frac{\beta}{1+\gamma\beta})} \tag{15}$$

$$\Gamma^2_{\mathrm{FLR}} = \Gamma^2_{\mathrm{GYR}} + \frac{k_y^2\lambda\delta\tau}{\Omega^2}\frac{p}{\rho}\frac{(1+\beta)(\tau\frac{p'}{\rho} - g)\frac{p'}{p} + [(1+\gamma\beta\tau)g - (1+\beta)\gamma\tau\frac{p'}{\rho}]\frac{\rho'}{\rho} + (\frac{\rho g}{p} - \tau\frac{p'}{p})g\beta}{(1+\gamma\beta)(1+\frac{k_y^2\delta^2\tau^2}{4\Omega^2}\frac{p}{\rho}\frac{\beta}{1+\gamma\beta})} \tag{16}$$

$$\Gamma^2_{\mathrm{GYR}} = \frac{\Gamma^2_{\mathrm{MHD}}}{1 + \frac{k_y^2\delta^2\tau^2}{4\Omega^2}\frac{p}{\rho}\frac{\beta}{1+\gamma\beta}} \tag{17}$$

$$\Gamma^2_{\mathrm{MHD}} = \frac{g^2}{u_A^2(1+\gamma\beta)} - \frac{\rho'}{\rho}g \tag{18}$$

$$D_{\mathrm{FLR}} = -\frac{k_y\lambda}{\Omega}\frac{\frac{\rho'}{\rho}g\tau\frac{p}{\rho}(\ln\frac{p}{\rho^\gamma})'}{(1+\gamma\beta)(1+\frac{k_y^2\delta^2\tau^2}{4\Omega^2}\frac{p}{\rho}\frac{\beta}{1+\gamma\beta})}. \tag{19}$$

Here, $\beta = \mu_0 p/B^2$, $u_A^2 = B^2/\mu_0\rho$, and $A' = dA/dx$. In obtaining Eq. (14), no assumption about the ordering of ω is explicitly made. This dispersion relation is applicable in finite-β and general nonisentropic thermal plasma regimes. It recovers the dispersion relation in [5] for an isentropic plasma where $(p/\rho)' = 0$ in the limit of zero β. Whereas the extended MHD model is only strictly valid in the small Larmor radius regime where $(\lambda, \delta) \ll 1$, the above dispersion relation is formally applicable in regimes where $(\lambda, \delta) \sim 1$ as well. Such a feature allows the benchmarking of extended MHD simulations in a wide range of parameter regimes. A similar dispersion relation was obtained by Ferraro and Jardin [7] for an isothermal plasma in the low frequency regime. The dispersion relation above applies to general plasma equilibrium, including the isothermal case. However, as shown in the following, the FLR stabilization properties are sensitive to the equilibrium considered.

Formally setting $\lambda = 0$, $\delta = 1$, the local dispersion relation for the pure interchange g mode reduces to the following form with FLR effects due only to ion gyroviscous force in momentum equation:

$$\omega^2 + \omega_{*\mathrm{GYR}}\omega + \Gamma^2_{\mathrm{GYR}} = 0, \tag{20}$$

where

$$\omega_{*\mathrm{GYR}} = \frac{\frac{k_y\tau}{\Omega}[(1+\beta)\frac{p'}{\rho} - \frac{2+\gamma\beta}{1+\gamma\beta}g\beta]}{1 + \frac{k_y^2\tau^2}{4\Omega^2}\frac{p}{\rho}\frac{\beta}{1+\gamma\beta}}. \tag{21}$$

Complete FLR stabilization (no unstable roots) requires

$$\frac{k_y^2\tau^2}{\Omega^2} \geq \frac{k_c^2\tau^2}{\Omega^2} = \frac{4\Gamma^2_{\mathrm{MHD}}}{[(1+\beta)\frac{p'}{\rho} - \frac{2+\gamma\beta}{1+\gamma\beta}g\beta]^2 - \frac{p}{\rho}\frac{\beta}{1+\gamma\beta}\Gamma^2_{\mathrm{MHD}}}, \tag{22}$$

where k_c is the cutoff wave number. When $\beta \to 0$, there is always a k_c for complete FLR stabilization by ion gyroviscosity. This is the results given in [5]. For finite β, there is a possibility that a real value of k_c may not exist because of the negative sign in front of the second term in the denominator of the expression for k_c in (22). For the isothermal plasma equilibrium studied by Ferraro and Jardin [7], $p' = p\rho'/\rho$, and the cutoff wave number k_c^{FJ} is given by

$$\left(\frac{k_c^{\mathrm{FJ}}\tau\beta}{\Omega}\right)^2 = \frac{4\Gamma^2_{\mathrm{MHD}}}{[\frac{u_A^2}{L_\rho}(1+\beta) + \frac{2+\gamma\beta}{1+\gamma\beta}g]^2 - \frac{u_A^2}{1+\gamma\beta}\Gamma^2_{\mathrm{MHD}}}, \tag{23}$$

where $L_\rho = -(d\ln\rho/dx)^{-1}$ and is assumed to be independent of β. It can be shown that for $g/L_\rho > 0$, the right-hand side of (23) is positive definite. A real value of k_c^{FJ} exists that completely stabilizes the g mode by FLR for isothermal equilibria in all regimes of β. This is consistent with the findings in [7].

For the equilibrium where the magnetic field is uniform, as was studied by Schnack and Kruger [10], $p' = \rho g$, so that the cutoff wave number k_c^{SK} is determined by

$$\left(\frac{k_c^{\mathrm{SK}}\tau}{\Omega}\right)^2 = \frac{4(1+\gamma\beta)\Gamma^2_{\mathrm{MHD}}}{u_A^2\frac{g}{L_\rho}(\beta_- - \beta)(\beta_+ + \beta)}, \tag{24}$$

where

$$\beta_\pm = \frac{\sqrt{(2-\gamma)^2 g^4 + \frac{4u_A^2 g^3}{L_\rho}} \pm (2-\gamma)g^2}{2u_A^2\frac{g}{L_\rho}}. \tag{25}$$

When $g/L_\rho > 0$ the denominator in the right-hand side of (24) is a monotonically decreasing function of β (for $\beta > 0$), and becomes zero and negative when $\beta \geq \beta_{\mathrm{crit}} = \beta_-$. As it turns out, for the particular equilibrium case studied by Schnack and Kruger [10] in NIMROD simulations, $\beta_{\rm crit} \sim 0.45$, whereas in the center of the simulation domain $\beta \sim 0.5$.

For the same type of equilibria with a uniform magnetic field, we further performed a detailed comparison between the NIMROD simulations and the dispersion relation in (20). The equilibrium is set up as follows: $\rho(x) = \rho_0 \exp(-x/L_\rho)$, $\mathbf{B}(x) = B_0\mathbf{e}_z$, $p(x) = \beta_0 B_0^2/\mu_0 + \rho_0 g L_\rho[1-\exp(-x/L_\rho)]$, where ρ_0, B_0, and β_0 represent the values of the corresponding quantities at the center of the simulation domain ($x = 0$). Following the earlier simulation study in [10], the particular choice of parameters is $\rho_0 = 6.689 \times 10^{-7}$, $B_0 = 6$, $L_\rho = 10$, $g = 10^{12}$, and $\tau = 1/2$. All physical quantities and parameters in this Letter are in SI units. A set of equilibria are further specified by a range of β_0 values from 0.1 to 0.5. For each β_0, the equilibrium is perturbed with a pure interchange motion as in (13) at $t = 0$. The MHD equations are allowed to evolve linearly until an exponentially growing mode is obtained and its converged growth rate can be measured. We compare the measured growth rate with the growth rate evaluated from the analytical dispersion in (20) and (21) at $x = 0$ where $\beta = \beta_0$.

The results are shown in Fig. 1. The $\beta = 0$ case corresponds to the dispersion relation given by Roberts and Taylor [5]. There is no simulation data for this case, since simulations of g mode with strictly zero pressure is numerically onerous. The $\beta = 0.5$ case is for the equilibrium originally studied by Schnack and Kruger [10]. In this case, the ion gyroradius $d_i \simeq 0.01$ at the center of the simulation domain. This case is qualitatively different from all other cases in the figure, in that $\beta > \beta_{\rm crit} \sim 0.45$, and therefore there is no real value of the cutoff k_y. Theory and simulation have a reasonable agreement except very near marginal stability. In that situation, the growth rate and the cutoff wave number obtained from simulations tend to be larger than the analytic values.

In the cases shown in Fig. 1, the cutoff wavelengths at FLR stabilization reach the order of ion gyroradius d_i, or $k_c d_i \gtrsim 1$. However, the absence of FLR stabilization due to finite β is also effective in the regime $k_y d_i < 1$, where the extended MHD model is physically relevant. One such example is shown in Fig. 2, where $L_\rho = 100$, $g = 10^9$, while other equilibrium specifications (except β) remain the same as in the earlier case. The reduced growth rate allows the FLR stabilization to first occur within the regime of $k_y d_i < 1$. Here in Fig. 2, the squares of the linear growth rates (Γ^2) are plotted as a function of $k_y d_i$, for equilibria with a different set of increasing β values. The lines are obtained from analytical dispersion relation in (20) and (21), and the symbols are measured from simulations. The FLR stabilization first starts at very low $\beta = 3 \times 10^{-5}$ and a small $k_y d_i \sim 0.33$. As β becomes larger, the cutoff value of $k_c d_i$ quickly grows beyond the $k_y d_i < 1$ regime and approaches infinity when β is raised close to about 5%. The $\beta = 5\%$ case is where the complete FLR stabilization is absent as the cutoff k_y becomes imaginary. But as shown here, even within the physically relevant regime of $k_y d_i < 1$, a very small increase of β is enough for the loss of FLR stabilization, as indicated by the $\beta = 0.001$ case where the cutoff k_y falls outside the regime of $k_y d_i < 1$. Thus this effect is both formally and physically relevant.

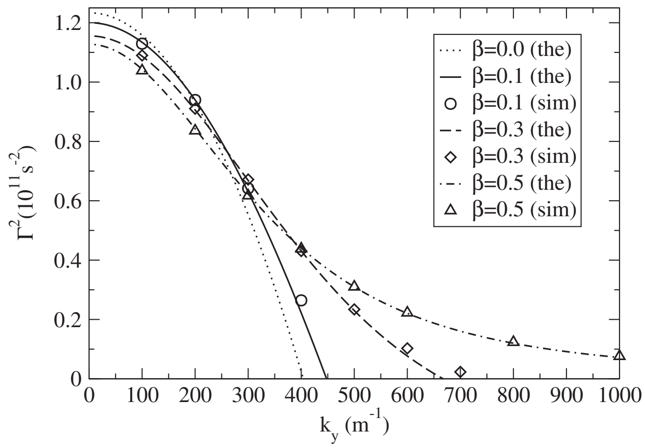

FIG. 1. The squares of the linear growth rates $\Gamma^2(10^{11}\ \mathrm{s}^{-2})$ [where $\Gamma = \mathrm{Im}(\omega)$] as a function of k_y (m^{-1}) for equilibria with an increasing set of β values. Here $L_\rho = 10$ m, $g = 10^{12}$ m/s^2. The lines are calculated from analytical dispersion in (20) and (21) (denoted as "the"), and the symbols are measured from simulations (denoted as "sim").

In the case when only the two-fluid effects are included by the generalized Ohm's law whereas the gyroviscosity is ignored, the local dispersion relation for the pure interchange g mode can be obtained as follows by formally setting $\lambda = 1$, $\delta = 0$ in (14):

$$\omega(\omega^2 + \omega_{*2\mathrm{FL}}\omega + \Gamma_{\mathrm{MHD}}^2) + D_{2\mathrm{FL}} = 0, \qquad (26)$$

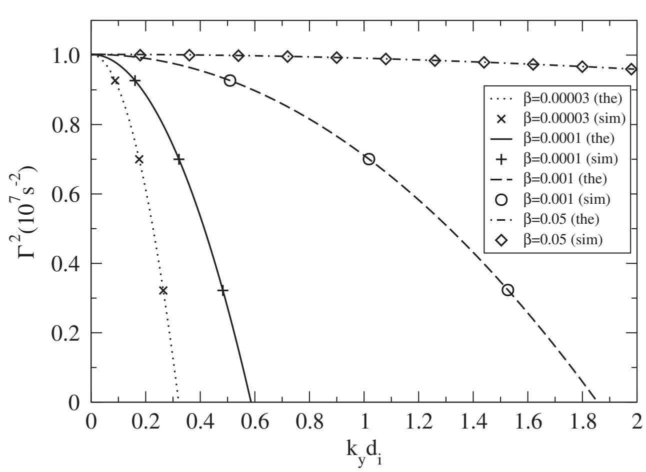

FIG. 2. The squares of the linear growth rates $\Gamma^2(10^7\ \mathrm{s}^{-2})$ [where $\Gamma = \mathrm{Im}(\omega)$] as a function of $k_y d_i$ for equilibria with an increasing set of β values. Here $L_\rho = 100$ m, $g = 10^9$ m/s^2. The rest are the same as in Fig. 1.

where

$$\omega_{*2\mathrm{FL}} = -\frac{k_y}{\Omega}\frac{1}{1+\gamma\beta}\left[g - \tau\frac{p}{\rho}\left(\ln\frac{p}{\rho^\gamma}\right)'\right] \tag{27}$$

$$D_{2\mathrm{FL}} = -\frac{k_y}{\Omega}\frac{\frac{\rho'}{\rho}g}{1+\gamma\beta}\tau\frac{p}{\rho}\left(\ln\frac{p}{\rho^\gamma}\right)'. \tag{28}$$

(Here the subscript "2FL" stands for "two-fluid.") The above dispersion relation reduces to that in [5] in an isentropic plasma where the entropy density $\ln(p/\rho^\gamma)$ is a constant. For nonisentropic plasma where $D_{2\mathrm{FL}} \neq 0$, there are 3 eigenmodes. When $(g/L_\rho)/(\omega\Omega)$ or $D_{2\mathrm{FL}}/\omega$ is not very small, there are situations when there are 2 complex conjugate roots so that there is always one growing mode for any k_y. When that happens, complete FLR stabilization could fail.

In the low frequency or weakly unstable regime, where $(g/L_\rho)/(\omega\Omega) \ll 1$ so that $D_{2\mathrm{FL}}/\omega \sim 0$, the complete FLR stabilization criterion is simply $\omega_{*2\mathrm{FL}}^2 > 4\gamma_{\mathrm{MHD}}^2$, or

$$\frac{k_y^2}{\Omega^2} \geq \frac{k_c^2}{\Omega^2} = \frac{4(1+\gamma\beta)^2\Gamma_{\mathrm{MHD}}^2}{[g - \tau\frac{p}{\rho}(\ln\frac{p}{\rho^\gamma})']^2}. \tag{29}$$

Again it is also possible to find an equilibrium such that the denominator in the expression for the cutoff wave number in (29) becomes identical or close to zero, so that the complete FLR stabilization effects could be entirely lost.

In summary, our simulations and theoretical calculations indicate that complete FLR stabilization of the interchange mode may not be attainable by ion gyroviscosity or the two-fluid effect alone in the framework of extended MHD. For a class of plasma equilibria in certain finite-β or nonisentropic regimes, the critical wave number for complete FLR stabilization tends toward infinity. The FLR stabilization of g mode with high wave number may not appear to be ubiquitous as is generally thought.

The result that FLR stabilization is incomplete or absent in certain regimes of β and equilibria is independent of the model used (either kinetic or two-fluid) when the conditions $k_y d_i \ll 1$ and $\beta \leq 1$ are met. In this regime, the corresponding kinetic theory should give the same results as the two-fluid model presented here since the two-fluid model is a valid moment representation of the kinetic model. In regimes where the two-fluid model is not physically valid, such as $k_y d_i \gg 1$, the question remains open as to how much the results will change, qualitatively or quantitatively, in a kinetic model with full FLR effects. This calculation is beyond the scope of the present work.

In this work, we focus our studies of incomplete FLR stabilization effects within the framework of the extended MHD emphasizing the physically valid regime of $k_y d_i \ll 1$. Such a study, though not fully kinetic, serves at least two purposes. First, it provides real physical insights in the physically valid regimes of the extended MHD model in general. Second, it provides a powerful means of verification for direct MHD simulation codes, such as NIMROD, in all mathematically valid FLR regimes of the extended MHD model. Our findings might also have potential implications for the extended MHD modeling and simulations of other interchange types of instabilities in magnetized plasmas [12–18].

This research is supported by U.S. DOE Grant No. DE-FG02-86ER53218, and by the Graduate School at University of Wisconsin-Madison. Computations were performed at NERSC, which is supported by U.S. DOE Contract No. DE-AC03-76SF00098. The authors are grateful for discussions with N. M. Ferraro and S. C. Jardin. P. Z. also thanks L. L. LoDestro for suggesting references to earlier work on FLR and for helpful discussions.

[1] M. N. Rosenbluth, N. A. Krall, and N. Rostoker, in *Proceedings of the Conference on Plasma Physics and Controlled Nuclear Fusion Research, Salzburg, Austria, 1961* (IAEA, Vienna, 1962).
[2] M. N. Rosenbluth and A. Simon, Phys. Fluids **8**, 1300 (1965).
[3] L. D. Pearlstein and N. A. Krall, Phys. Fluids **9**, 2231 (1966).
[4] E. Bowers and M. G. Haines, Phys. Fluids **11**, 2695 (1968).
[5] K. V. Roberts and J. B. Taylor, Phys. Rev. Lett. **8**, 197 (1962).
[6] J. D. Huba, Phys. Plasmas **3**, 2523 (1996).
[7] N. M. Ferraro and S. C. Jardin, Phys. Plasmas **13**, 092101 (2006).
[8] S. E. Kruger *et al.*, https://nimrodteam.org/.
[9] D. D. Schnack *et al.*, Phys. Plasmas **13**, 058103 (2006).
[10] D. D. Schnack and S. E. Kruger (private communication).
[11] C. R. Sovinec *et al.*, J. Phys. Conf. Ser. **16**, 25 (2005).
[12] E. Bowers and M. G. Haines, Phys. Fluids **14**, 165 (1971).
[13] R. J. Wright, D. F. R. Pott, and M. G. Haines, Plasma Phys. **18**, 1 (1976).
[14] J. P. Freidberg and L. D. Pearlstein, Phys. Fluids **21**, 1207 (1978).
[15] L. D. Pearlstein and J. P. Freidberg, Phys. Fluids **21**, 1218 (1978).
[16] W. A. Newcomb, Phys. Fluids **28**, 505 (1985).
[17] B. I. Cohen, R. P. Freis, and W. A. Newcomb, Phys. Fluids **29**, 1558 (1986).
[18] M. Z. Caponi, B. I. Cohen, and R. P. Freis, Phys. Fluids **30**, 1410 (1987).