

Real-Time Gauge/Gravity Duality

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We present a general prescription for the holographic computation of real-time n -point functions in nontrivial states. In quantum field theory such real-time computations involve a choice of a time contour in the complex time plane. The holographic prescription amounts to “filling in” this contour with bulk solutions: real segments of the contour are filled in with Lorentzian solutions while imaginary segments are filled in with Riemannian solutions and appropriate matching conditions are imposed at the corners of the contour. We illustrate the general discussion by computing the 2-point function of a scalar operator using this prescription and by showing that this leads to an unambiguous answer with the correct $i\epsilon$ insertions.

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The gravity/gauge theory duality has been one of the most far reaching developments in recent years. On the one hand it opens a window into strong coupling dynamics of gauge theories and on the other hand it provides a realization of holography and offers a new perspective in gravitational physics. In recent times, it has found applications that range from phenomenology to condensed matter physics.

The foundational papers on the subject [1] laid down the basic principles of the duality. The detailed dictionary between bulk and boundary physics, however, is best understood to date in the supergravity approximation and in the Euclidean regime; i.e., the bulk solution involves a hyperbolic Riemannian manifold and the boundary theory is Wick-rotated. While this suffices for many applications, there are also many reasons for developing a general real-time prescription. Such a real-time formalism should be used, for example, in studies of time-dependent phenomena, analysis of gauge theories in nontrivial pure or mixed states, or the holographic interpretation of nonstationary spacetimes.

Such a formalism, applicable at the same level of generality as the corresponding Euclidean prescription, would constitute an integral part of the definition of the holographic correspondence and as such is important on general grounds. Furthermore, there is an urgency for setting up such a formalism since interesting current applications, for example the holographic modeling of the quark-gluon plasma, crucially involve real-time physics. Actually much of the recent work on real-time holographic prescriptions was driven by such applications, see [2] for a review. The aim of this work is to provide a concrete, first principles prescription that covers all n -point functions and is applicable for any quantum field theory (QFT) that has [an asymptotically anti-de Sitter (AdS)] holographic dual. Previous work on this subject includes [3–5] and our results agree with these works when we restrict to their respective domains of validity.

The basic Euclidean holographic dictionary identifies the boundary conditions $\phi_{(0)}$ for the bulk fields Φ to

sources of the dual boundary operators and the bulk partition function, which is a functional of these boundary fields, to the generating functional of connected n -point functions. The main new issue that arises in the Lorentzian context is that in the bulk, on top of specifying boundary conditions $\phi_{(0)}$, one also needs to specify initial and/or final conditions ϕ_{\pm} for all fields, and the bulk partition function is also a functional of these. The main question is to understand their meaning in the dual QFT. Intuitively, ϕ_{\pm} should be related to QFT in- and out-states [6], but an exact prescription to translate QFT states to initial and final boundary data for the bulk fields has not previously been worked out.

Let us briefly recall some QFT basics that are relevant to our discussion. Consider a field configuration with initial condition $\phi_{-}(\vec{x})$ at $t = -T$ and final condition $\phi_{+}(\vec{x})$ at $t = T$. The path integral with fields constrained to satisfy these conditions produces the transition amplitude $\langle \phi_{+}, T | \phi_{-}, -T \rangle$. If we are interested in vacuum amplitudes we should multiply this expression by the vacuum wave function $\langle 0 | \phi_{+}, T \rangle$ and $\langle \phi_{-}, -T | 0 \rangle$ and integrate over ϕ_{+} , ϕ_{-} . The insertion of these wave functions is equivalent to extending the fields in the path integral to live along a contour in the complex time plane as sketched in Fig. 1. Indeed, the infinite vertical segment starting at $-T$ corresponds to a transition amplitude $\lim_{\beta \rightarrow \infty} \langle \phi_{-}, -T | e^{-\beta H} | \Psi \rangle$ for some state $|\Psi\rangle$, which is however irrelevant since taking the limit projects it onto the vacuum wave function $\langle \phi_{-}, -T | 0 \rangle$. Similarly, we obtain $\langle 0 | \phi_{+}, T \rangle$ from the vertical segment starting at $t = T$.

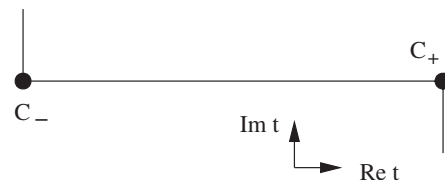


FIG. 1. The contour in the complex time plane used to obtain vacuum-vacuum amplitudes.

Recall also that these wave function insertions ultimately lead to the $i\epsilon$ factors in the Feynman propagators.

If one wants to compute expectation values in nontrivial states or thermal ensembles then one should consider different time contours, like a real-time thermal contour or a closed in-in contour.

Prescription.—The holographic prescription we propose is to use “piecewise” holography: for each real segment of the time contour we consider a Lorentzian solution and for each imaginary part an Euclidean solution. At the corners of the contour, the various bulk solutions are joined together using standard matching conditions; i.e., the induced values of the fields and their conjugate momenta should be (appropriately) continuous along the gluing surface, which is some hypersurface in the bulk. This results in a completely holographic prescription where all data are encoded in the conformal boundary of the entire spacetime; the initial and final states are encoded in the boundary of the Euclidean parts.

The next step is to compute the value of the combined (Euclidean plus Lorentzian) on-shell actions and then vary these with respect to sources to obtain the renormalized holographic 1-point functions in the presence of sources. This results in a formula that relates the 1-point function to the asymptotics of the bulk solution [7]. Recall that the holographic renormalization required in this procedure relies on having sufficient control over the asymptotics of the bulk solutions. This analysis, however, is independent of the signature of the spacetime, so all standard results carry over immediately. One only needs to check that the corners where Lorentzian and Euclidean solutions are joined do not introduce any complications. The matching conditions ensure that this is the case, as will be described elsewhere [8].

As usual, to compute holographic n -point functions we need to solve the bulk field equations to order $(n - 1)$ around the bulk solution. The result should then be sub-

stituted in the $(n - 1)$ th variation of the holographic 1-point function to obtain the n -point function. Of course, for this procedure to be well posed, the solution to these bulk field equations, subject to boundary conditions as specified above, must be unique. On general grounds we expect that the prescription given here has this property, since we specify enough data, and we will also illustrate this in the first nontrivial example below.

Example.—We now illustrate our general discussion in the simplest possible setup. Namely, we will discuss the duality for the contour of Fig. 1, and compute a conformal field theory (CFT) two-point function to show that we find the correct $i\epsilon$ insertions. A more extended discussion that includes a discussion of examples corresponding to other time contours (thermal, closed time, etc.), fields, Asymptotically AdS spacetimes, etc., will be presented in [8].

As discussed above, the contour in Fig. 1 corresponds to real-time vacuum-to-vacuum correlators. Although the discussion can be easily done for any CFT_d (with a holographic dual), for concreteness we specialize here to $d = 2$. Including the spatial S^1 direction, we have redrawn the contour in Fig. 2, and we have also compactified the Euclidean semi-infinite cylinders by adding a point at infinity. The corners of the contour are two circles which we denote as C_{\pm} . The prescription now amounts to holographically filling in this surface with a bulk manifold consisting of three components, namely, a segment M_L of Lorentzian AdS_3 and two “caps” M_{\pm} consisting of half of Euclidean AdS_3 . One can view these caps as providing a Hartle-Hawking wave function on the hypersurfaces S_{\pm} (where $\partial S_{\pm} = C_{\pm}$). In this respect, our prescription is not only field theory inspired but also in line with standard considerations on wave functions in quantum gravity [9]; see [5,10] for a related discussion in the context of AdS/CFT.

We now propose that the relation between bulk and boundary quantities reads

$$\langle 0|T \exp\left(-i \int_{\delta M_L} d^d x \sqrt{-g} \phi_{(0)} \mathcal{O}\right)|0\rangle = \exp(iI_L[\phi_{(0)}, \phi_-, \phi_+] - I_E[0, \phi_-] - I_E[0, \phi_+]) \quad (1)$$

with δM_L the conformal boundary of M_L as in Fig. 2, $I_L[\phi_{(0)}, \phi_-, \phi_+]$ the *on-shell* Lorentzian action for M_L that depends not only on $\phi_{(0)}$ but also on initial and final data ϕ_{\pm} , and $I_E[\phi_{(0,\pm)}, \phi_{\pm}]$ the Euclidean on-shell actions on the half-Euclidean spaces M_{\pm} with sources $\phi_{(0,\pm)}$ and boundary condition ϕ_{\pm} at S_{\pm} . In (1) we set the sources $\phi_{(0,\pm)}$ to zero since we are interested in vacuum-to-vacuum correlators. Nonzero values for $\phi_{(0,\pm)}$ would correspond to changing the initial and/or final state, as it does in the CFT. As the notation indicates, we expect (and this will be verified below) that this procedure leads to time-ordered products.

Finally, we need to fix ϕ_{\pm} by specifying the behavior of the solution at the corners. This we do by imposing the following two “matching conditions” for the fields across

the S_{\pm} : (i) As is already indicated in (1), we impose that the induced values of the bulk fields, so the ϕ_{\pm} , are the same

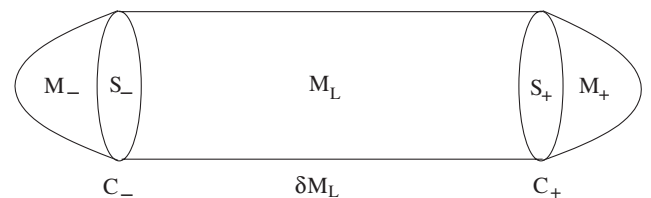
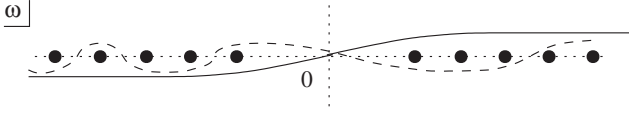


FIG. 2. The CFT_2 contour, with the spatial circle and points at infinity added. Our prescription is to fill it in with an AdS spacetime consisting of three parts as well. The corners C_{\pm} extend to hypersurfaces S_{\pm} in the bulk, and δM_L is the cylindrical conformal boundary of M_L .

FIG. 3. Contours around the poles in the complex ω plane.

on both sides of S_{\pm} . (ii) We also demand *stationarity* of the combined on-shell supergravity actions with respect to variations with respect to ϕ_{\pm} :

$$\frac{\delta}{\delta\phi_{\pm}}(iI_L[\phi_{(0)}, \phi_-, \phi_+] - I_E[\phi_{(0,-)}, \phi_-] - I_E[\phi_{(0,+)}, \phi_+]) = 0 \quad (2)$$

which should be read as an equation for ϕ_{\pm} .

Some comments are in order. First, since we think of the bulk solution as a saddle point to some stringy path integral, these conditions are a direct consequence of the saddle-point approximation. Second, taking derivatives of an on-shell action gives the conjugate momentum, so we obtain the standard junction conditions [11], except for extra factors of i . Last, the data ϕ_{\pm} should be compatible with $\phi_{(0)}$ and $\phi_{(0,\pm)}$ at the corners C_{\pm} . In the example below, we will see how this is done.

Two-point function.—We now specialize to a free massive scalar Φ , propagating without backreaction on empty AdS_3 , capped off with two Euclidean half-balls as in Fig. 2. Our aim is to holographically compute the two-point function of the operator \mathcal{O} dual to Φ , including the correct $i\epsilon$ terms, with the above prescription. The relevant part of the supergravity action is simply

$$S = \frac{1}{2} \int d^3x \sqrt{|G|} (-\partial_{\mu} \Phi \partial^{\mu} \Phi - m^2 \Phi^2). \quad (3)$$

The dimension of \mathcal{O} is $\Delta = 1 + \sqrt{1 + m^2} = 1 + l$ with $l \in \{0, 1, 2, \dots\}$.

First consider the scalar field solution in the Lorentzian spacetime without the caps. In the AdS_3 background,

$$ds^2 = -(r^2 + 1)dt^2 + \frac{dr^2}{r^2 + 1} + r^2 d\phi^2,$$

the mode solutions to the Klein-Gordon equation are of the form $e^{-i\omega t + ik\phi} f(\omega, \pm k, r)$ with

$$\begin{aligned} f(\omega, k, r) &= C_{\omega kl} (1 + r^2)^{\omega/2} r^k F(\hat{\omega}_{kl}, \hat{\omega}_{kl} - l; k + 1; -r^2) \\ &= r^{l-1} + \dots + r^{-l-1} \alpha(\omega, k, l) [\ln(r^2) \\ &\quad + \beta(\omega, k, l)] + \dots \end{aligned}$$

where $\hat{\omega}_{kl} = (\omega + k + 1 + l)/2$, $C_{\omega kl}$ is a normalization factor chosen such that the coefficient of the leading term equals 1 and in the last line we omitted terms of lower powers of r and some terms polynomial in ω and k (which would lead to contact terms in the 2-point function). Furthermore,

$$\begin{aligned} \alpha(\omega, k, l) &= (\hat{\omega}_{kl} - l)_i (\hat{\omega}_{kl} - k - l)_i / (l!(l-1)!), \\ \beta(\omega, k, l) &= -\psi(\hat{\omega}_{kl}) - \psi(\hat{\omega}_{kl} - \omega - l), \end{aligned} \quad (4)$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol and $\psi(x) = d \ln \Gamma(x)/dx$ is the digamma function. Note also that $f(\omega, k, r) = f(-\omega, k, r)$. Only the $f(\omega, k, r)$ with $k \geq 0$ are regular for $r \rightarrow 0$, so the modes we use below are of the form $e^{-i\omega t + ik\phi} f(\omega, |k|, r)$.

We would now like to obtain the most general solution whose leading asymptotics ($\sim r^{l-1}$ as $r \rightarrow \infty$) contain an arbitrary source $\phi_{(0)}(t, \phi)$ for the dual operator. This solution is

$$\begin{aligned} \Phi(t, \phi, r) &= \frac{1}{4\pi^2} \sum_{k \in \mathbb{Z}} \int_C d\omega \int d\hat{t} \int d\hat{\phi} e^{-i\omega(t-\hat{t}) + ik(\phi-\hat{\phi})} \\ &\quad \times \phi_{(0)}(\hat{t}, \hat{\phi}) f(\omega, |k|, r) \\ &\quad + \sum_{\pm} \sum_{k \in \mathbb{Z}} \sum_{n=0}^{\infty} c_{nk}^{\pm} e^{-i\omega_{nk}^{\pm} t + ik\phi} g(\omega_{nk}, |k|, r), \end{aligned} \quad (5)$$

which we now proceed to explain. Clearly, the first term in (5) has the correct leading behavior, but we should explain the C representing a contour in the complex ω plane. This contour is necessary to avoid the poles in $\alpha(\omega, k, l)\beta(\omega, k, l)$ at

$$\omega = \omega_{nk}^{\pm} \equiv \pm(2n + k + 1 + l), \quad n \in \{0, 1, 2, \dots\}.$$

We are now completely free to specify any contour that circumvents the poles, for example, the striped contour in Fig. 3. The difference between two contours is a sum over the residues:

$$\begin{aligned} g(\omega_{nk}, k, r) &= \oint_{\omega_{nk}} d\omega f(\omega_{nk}, k, r) \sim r^{-l-1} \alpha(\omega_{nk}, k, l) \\ &\quad \times \left(\oint_{\omega_{nk}} d\omega \beta(\omega, k, l) \right). \end{aligned}$$

The $g(\omega_{nk}, k, r)$ are the “normalizable modes.” Since they vanish asymptotically, we can actually freely add them to the solution Φ (so not just as residues) without affecting the fact that $\Phi \sim \phi_{(0)} r^{l-1}$ for large r . Therefore, the most general solution includes a sum over these normalizable modes with arbitrary coefficients c_{nk}^{\pm} , as appears in (5). Since a change of contour can be undone by also changing the c_{nk}^{\pm} , let us fix the contour to be the solid line in Fig. 3. This means all the nonuniqueness in the Lorentzian solution is captured by the c_{nk}^{\pm} .

For later use, let us present an alternative form of the solution. Without loss of generality, we can assume that the initial matching surface S_- is at $t = 0$ and that the sources are zero in the vicinity of S_- . Then, near S_- , we can perform the ω integral by closing the contour and picking up the poles in $f(\omega, k, r)$, resulting in

$$\Phi = \frac{1}{4\pi^2} \sum_{n=0}^{\infty} \sum_{k \in \mathbb{Z}} e^{-i\omega_{nk}^- t + ik\phi} \phi_{(0)}(\omega_{nk}^-, k) g(\omega_{nk}, |k|, r) \\ + \sum_{\pm} \sum_{k \in \mathbb{Z}} \sum_{n=0}^{\infty} c_{nk}^{\pm} e^{-i\omega_{nk}^{\pm} t + ik\phi} g(\omega_{nk}, |k|, r),$$

where we Fourier transformed the source. Of course, this is an expected result; it just represents the completeness of the modes.

Now consider the solution on the “initial cap,” so on the space specified by the metric,

$$ds^2 = (r^2 + 1)d\tau^2 + \frac{dr^2}{r^2 + 1} + r^2 d\phi^2$$

with $-\infty < \tau \leq 0$, so that we have half of Euclidean AdS space. Had the bulk been the entire Euclidean AdS space, the Klein-Gordon equation would have a unique regular solution given boundary data. In particular, with zero sources the unique regular solution is identically equal to zero. In our case the sources are zero but we only consider half of the space, so solutions that would be excluded are now allowed because they are only singular at the other half of the space. These regular solutions are precisely the analytically continued Lorentzian normalizable modes, so we find solutions when $\omega = \omega_{nk}^{\pm}$. Since the solution should vanish at $\tau \rightarrow -\infty$, the most general Euclidean solution contains only negative frequencies,

$$\Phi(\tau, \phi, r) = \sum_{n,k} d_{nk}^- e^{-\omega_{nk}^- \tau + ik\phi} g(\omega_{nk}, |k|, r),$$

with thus far arbitrary coefficients d_{nk}^- .

We can now consider the matching at $\tau = t = 0$, which will fix the initial data. From the continuity $\Phi_L(0, \phi, r) = \Phi_E(0, \phi, r)$ we find, using orthogonality and completeness of the $g(\omega_{nk}, |k|, r)$:

$$\phi_{(0)}(\omega_{nk}^-, k) + c_{nk}^- + c_{nk}^+ = d_{nk}^-.$$

Equation (2) yields a relation between conjugate momenta,

$$-i\partial_t \Phi_L = \partial_\tau \Phi_E.$$

Substituting the solutions we find

$$-\omega_{nk}^- \phi_{(0)}(\omega_{nk}^-, k) - \omega_{nk}^- c_{nk}^- - \omega_{nk}^+ c_{nk}^+ = -\omega_{nk}^- d_{nk}^-$$

so that $c_{nk}^+ = 0$. Similarly, the matching to the out state determines $c_{nk}^- = 0$, and indeed all the freedom in the bulk solution is fixed. We remark that, had we chosen any other contour in (4), we would have found nonzero values of some of the c_{nk}^{\pm} , effectively throwing us back to the solid line of Fig. 3.

Finally, the two-point function is obtained from the r^{-l-1} term in the asymptotic expansion of (4) (with $c_{nk}^{\pm} = 0$):

$$\langle 0|T\mathcal{O}(t, \phi)\mathcal{O}(0, 0)|0\rangle = \frac{l}{4\pi^2 i} \sum_k \int_C d\omega e^{-i\omega t + ik\phi} \\ \times \alpha(\omega, |k|, l)\beta(\omega, |k|, l),$$

with the contour C being the same as for the bulk solution, thus the standard Feynman prescription leading to time-ordered correlators. We emphasize again that C was completely fixed by the matching to the caps. Integrating over C is equivalent to integrating over the real axis and shifting $\omega \rightarrow \omega(1 + i\epsilon)$. The Fourier transform of this expression then gives

$$\langle 0|T\mathcal{O}(t, \phi)\mathcal{O}(0, 0)|0\rangle = \frac{l^2/(2^{l+1}\pi)}{[\cos(t - i\epsilon t) - \cos(\phi)]^{l+1}}.$$

This is the expected form for a time-ordered two-point function on a cylinder and the normalization coefficient can be shown to agree with the standard AdS/CFT normalization of 2-point functions.

Conclusion.—We presented a prescription that relates in- and out-states of the boundary QFT to initial and final data for the bulk fields. We discussed in detail the case of a free bulk scalar field in pure AdS, but the procedure extends to other contours, fields (including the metric), asymptotically AdS spacetimes and higher n -point functions in a clear manner, details of which will be presented in [8]. The prescription allows us to study holographically QFT dynamics in cases where analytic continuation from the Euclidean regime does not suffice. It also offers a new perspective on the holographic encoding of bulk spacetimes, since the state or density matrix corresponding to a given geometry is directly related to the Euclidean parts of the solution. This may allow us to understand how regions beyond bulk horizons are “encoded” in the QFT data. We hope to address this and other intriguing aspects of real-time gauge/gravity duality in the near future.

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