Doing Arithmetic With Nonlinear Acoustic Vortices

Régis Marchiano*

Université Pierre et Marie Curie-Paris6, CNRS, UMR 7190, Institut Jean le Rond d'Alembert, 4 place Jussieu, Paris, F-75005 France

Jean-Louis Thomas

CNRS, Université Pierre et Marie Curie-Paris6, CNRS, UMR 7588, Institut des NanoSciences de Paris, 140 rue de Lourmel, Paris, F-75015 France

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Phase singularities of wave-front-like screw dislocations or vortices possess a well-defined quantity that can only take integer value: the topological charge. In the nonlinear regime, it has been demonstrated that optical or acoustical vortices interact and the topological charge follows a conservation law. Here this facility is used in nonlinear parametric interaction of two vortices shifted in frequency to perform sums and subtractions of the topological charge. Thus, we experimentally demonstrate a new technique to perform wave computation in the group of integer $\mathbb{Z}$. When the two vortices have commensurable frequencies, different combinations give the same frequencies but different tolopological charges may occur. We show that an energy criterion can be used to predict the outcome. A corollary is that a modulation of amplitude of the vortices switches from one result to the other.

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Waves can have a singularity of amplitude, phase, or polarization [1]. We are interested here in the singularities of phase and more particularly in the screw dislocations or vortices (optical or acoustical) [2]. In the latter case, a cut of the wave in a plane perpendicular to its axis of propagation shows a phase growing in a linear way from 0 to $2\pi m$ on a contour surrounding the axis of the vortex. m is called the charge of the vortex and results in a quantified orbital momentum [3,4]. This property was used to rotate absorbing particles within the framework of optical tweezers [5]. Applications to the trapping of an atom were also considered [6].

These vortices also exist for acoustic wave in linear and nonlinear regimes [7,8]. Optic, acoustic, and quantum mechanic equations are formally equivalent. So, acoustics can be used to study some special features of these fields of physics especially experimentally. Recently this property was used to produce and measure the torque exerted on a torsion pendulum by an acoustic wave [9,10]. Another interesting property of vortices is the storage and transmission of digitized information with the particularity that here the phase coding is in space and not in time [11–15]. The main interest in vortices for this kind of applications is the possibility of working with high order charge and thus increase the transmission capacity compared to a binary code. The synthesis of a vortex of high order is made thanks to space sampling of the field in optics (for instance with forked diffraction grating [16], holograms [17], or spatial light modulator (SLM) [18]) or in acoustics (by array of piezoelectric transducers [8]). The interest of this prospect to the transmission of information is reinforced by the robustness and the self-reconstruction capability of these waves during the propagation in a heterogeneous medium or after partial obstruction. In a nonlinear regime, it has been demonstrated that optical or acoustical vortices (AV) interact and the topological charge follows a conservation law [8,19,20]. For four waves mixing in the case of a cubic nonlinearity, the same nonlinear transfer of angular momentum is observed with two waves carrying angular momentum [21] or three waves [22]. In this work we show new advantages of these waves for the handling of information in nonlinear mode.

This Letter is divided into three parts. The first part is a description of the experimental setup. The second part is a study of the case where the two primary waves have incommensurable frequencies (then the states are not degenerate). The third part investigates the case of commensurable frequencies leading to degenerate states. It is shown how one can pass from one state to another and change the quantified information dynamically only by modulating the amplitude of initial states.

The synthesis of AV can be made in different ways: direct generation from a special shaped source [7], opto-acoustic generation [23], mode conversion from wave plane to AV [9], or with a technique of wave field synthesis such as inverse filter technique [8,24]. The latter technique allows an optimal control of the synthesized pressure field especially for the spatial distribution and the phase. This technique requires a set of control points and a set of sources. A first measurement of the impulsional responses of each source for all the control points allows one to know the linear operator linking the sources and control points. In a second step, the operator is numerically inversed in order to determine the signals to emit from the sources to produce a desired pattern on the control point. The synthesized field is then the best field (least squares sense) able to be produced by the sources. For the experiments presented here, the acoustical sources are 61 piezoelectrical transducers located on a flat hexagon. Experiments are made in a water tank (Fig. 1). For acoustical waves, water is a

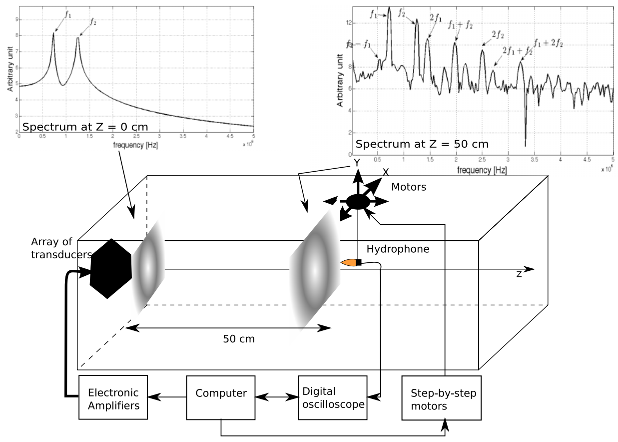

FIG. 1 (color online). Experimental setup.

quasinondispersive media. Each transducer is a disk of 0.5 mm radius working at the central frequency 1 MHz. The bandwidth of transducers is about 1 MHz; this property is used to produce vortices with different frequencies in the range 0.5–1.5 MHz. The set of control points is a flat square grid of 2916 control points equally spaced (one point every 1.5 mm). It is parallel to the plane containing the sources, and located 50 cm away from it. The transducers are driven independently from one another by a multichannel electronic device. The delivered electrical power on each channel can rise up to 50 W; thus, the transducers can emit acoustical waves with either weak amplitude (linear propagation) or high amplitude (nonlinear propagation). The instantaneous pressure field on the control point is measured by a needle hydrophone with a 15 MHz bandwidth. The hydrophone can be moved through the water tank by step-by-step motors (precision of the displacement is around 10 μm). Gauss-Laguerre beams are known to carry screw dislocation. They are characterized by two integers (l, m), l is the radial index and m is the topological charge discussed above. In this study, attention is paid only to the cases where $l = 0$. As the paraxial wave equation is formally the same in acoustics and quantum mechanics, we will use a formalism based on different states of the beam. We denote by $|f_i\rangle$ the state of the beam at frequency f_i. Operator $\mathbf{F}$ gives the frequency of the state: $\mathbf{F}|f_i\rangle = f_i$, and operator $\mathbf{M}$ gives the topological charge of the beam at frequency f_i: $\mathbf{M}|f_i\rangle = m_i$.

The experimental setup is used to produce the superposition of the states $|f_1\rangle$ and $|f_2\rangle$ with ($\mathbf{F}|f_1\rangle = 0.725$ MHz, $\mathbf{M}|f_1\rangle = 1$) and ($\mathbf{F}|f_2\rangle = 1.25$ MHz, $\mathbf{M}|f_2\rangle = 1$). The states $|f_1\rangle$ and $|f_2\rangle$ have the same axis. The desired field is then two coaxial acoustical vortices with slightly shifted frequencies. The expected dynamic is essentially nonlinear (see [25] for the arithmetic of noncoaxial vortices in a linear regime). As the signals computed by inverse filter technique are emitted with low amplitude, the propagation is linear and the measured states in the control points are $|f_1\rangle$ and $|f_2\rangle$. These results are not shown here. If the signals are sent with a higher amplitude, nonlinear effects affect the propagation. In fluids, the nonlinear effects for acoustical waves are quadratic and proportional to the instantaneous pressure [26]. They engender a nonlinear interaction between two collinear beams with slightly shifted frequencies known as parametric interaction [27]. Starting from frequencies f_1 and f_2, the parametric interaction or two waves mixing, engenders the creation of all the linear combinations of these quantities: interaction $[(f_1, f_2) \rightarrow f_2 - f_1, f_1 + f_2, 2f_1, 2f_2, \ldots]$. In the experiments presented here, this classical effect is well recovered as shown by the spectrum of the pressure field in the control plane (Fig. 1).

Acoustics allows one to measure the instantaneous pressure field (at least in the range of frequencies used in this experimental setup). This means that amplitude and phase are directly accessible. Figure 2 shows the instantaneous pressure field and the rms pressure field. The first one displays the pressure for a given time (selected randomly in the useful part of the signal). Two structures are clearly visible: one positive plus one negative lobe. These two lobes turn around each other in one period [24]. Figure 2(b) shows the root mean square pressure field. It displays the classical "doughnut" shape well known in optics [2].

Each subfigure of Fig. 3 displays the phase pattern at a frequency generated by the parametric interaction. The total topological charge can be found by counting the number of jumps of the phase (sharp transitions between white and black) along a close contour surrounding the singularity. The sign of the topological charge corresponds to the direction of rotation of the phase. The convention is positive if the rotation is anticlockwise to have positive jumps of the phase around the singularity and negative otherwise. Figure 3 presents the phase in the control plane corresponding to the first eight linear combinations of primary frequencies. The phase patterns at f_1 and f_2 conform to the desired field: $\mathbf{M}|f_1\rangle = m_1 = 1$ and $\mathbf{M}|f_2\rangle = m_2 = 1$. As already noted, nonlinear effects do not destroy or destabilize acoustical vortices [28]. For the generated frequencies, it is noticeable that their phase shapes possess a well-defined topological charge. Experimental results show the following topological charges: 0 for $\mathbf{F}|f_2 - f_1\rangle$,

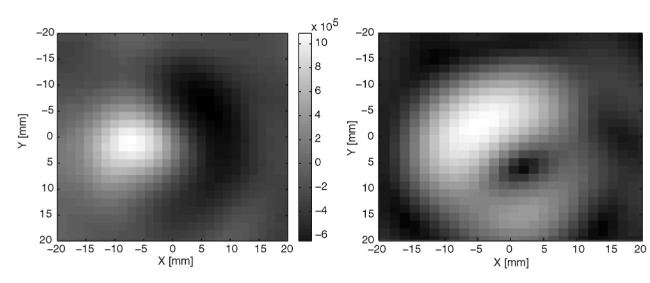

FIG. 2. Amplitude of the pressure field for configuration 1 (states ($\mathbf{F}|f_1\rangle = 0.725$ MHz, $\mathbf{M}|f_1\rangle = 1$) and ($\mathbf{F}|f_2\rangle = 1.25$ MHz, $\mathbf{M}|f_2\rangle = 1$)). The left-hand figure is the instantaneous pressure and the right-hand figure is the rms pressure.

1 for $\mathbf{F}|f_1\rangle$, 1 for $\mathbf{F}|f_2\rangle$, 2 for $\mathbf{F}|2f_1\rangle$, 2 for $\mathbf{F}|2f_2\rangle$, 2 for $\mathbf{F}|f_1+f_2\rangle$, 3 for $\mathbf{F}|2f_1+f_2\rangle$, and 3 for $\mathbf{F}|f_1+2f_2\rangle$. It is possible to link these topological charges to those of the primary frequencies: the topological charge at frequency pf_1+qf_2, with $(p,q)\in\mathbb{N}$, can be calculated by applying the same linear combination on the primary topological charges:

$$\mathbf{M}\,|pf_1+qf_2\rangle = pm_1+qm_2. \qquad (1)$$

This rule works for all configurations presented in Fig. 3 even for the null charge [Fig. 3(a) for the frequencies difference]. This rule can be viewed as a generalization of previous theoretical works [29] derived in the case of the sum frequency mixing (only for $p=q=1$; note that the two studies can be compared only in the case of no walk-off between the primary waves). The case of sum frequency mixing has also been used to produce solitons. It has been observed in optics [30] that, at high intensity, three wave mixing results in solitons break up induced by an azimuthal instability, and the number of solitons is related to the topological charge of primary waves. This is an example of another way of doing arithmetic with vortices This rule, Eq. (1), permits one to build arithmetics in $\mathbb{Z}$; indeed, it is also valid for negative signs of the topological charge. Figure 4 shows the first eight patterns of the phase in the control plane for two initial states: ($\mathbf{F}|f_1\rangle=0.95$ MHz, $\mathbf{M}|f_1\rangle=2$) and ($\mathbf{F}|f_2\rangle=1.65$ MHz, $\mathbf{M}|f_2\rangle=1$). Experimental results in Fig. 4 show the following topological charges: -1 for $\mathbf{F}|f_2-f_1\rangle$, 2 for $\mathbf{F}|f_1\rangle$, 1 for $\mathbf{F}|f_2\rangle$, 4 for $\mathbf{F}|2f_1\rangle$, 2 for $\mathbf{F}|2f_2\rangle$, 3 for $\mathbf{F}|f_1+f_2\rangle$, 5 for $\mathbf{F}|2f_1+f_2\rangle$, and 4 for $\mathbf{F}|f_1+2f_2\rangle$. The phase pattern at the frequency difference is -1. Note that this is an example of inversion of the topological charge already observed in optics or acoustics for a propagation through cylindrical lenses [31–33].

Nevertheless, the rule given in Eq. (1) is not valid if primary frequencies are commensurable. Indeed, different combinations of frequencies can give the same result but not necessarily the same topological charges. Figure 5 illustrates this situation for initial states ($\mathbf{F}|f_1\rangle=1$ MHz, $\mathbf{M}|f_1\rangle=1$) and ($\mathbf{F}|f_2\rangle=1.5$ MHz, $\mathbf{M}|f_2\rangle=1$). For this configuration, $F|3f_1\rangle=F|2f_2\rangle=3$ MHz, but applying the rule [Eq. (1)] leads to nonsense as $3m_1\neq 2m_2$. The experimental results show that for this configuration the state is $\mathbf{M}|3f_1\rangle=\mathbf{M}|2f_2\rangle=3m_1=3$. To fix that problem an energetic criterion can be introduced in addition to the arithmetic rule (1). The energies of the degenerate modes are noted $E_{pq}=\mathbf{E}|pf_1+qf_2\rangle$ and $E_{rs}=\mathbf{E}|rf_1+sf_2\rangle$, where $\mathbf{E}$ is the energy operator giving the energy associated with a state. In the case of degenerate states $\mathbf{F}|pf_1+qf_2\rangle=\mathbf{F}|rf_1+sf_2\rangle$, the topological charge can be found considering the state of maximum energy: if $E_{pq}>E_{rs}$ then $\mathbf{M}|pf_1+qf_2\rangle=\mathbf{M}|rf_1+sf_2\rangle=pm_1+qm_2$; otherwise, if $E_{pq}<E_{rs}$ then $\mathbf{M}|pf_1+qf_2\rangle=\mathbf{M}|rf_1+sf_2\rangle=rm_1+sm_2$. The last criterion shows that the amplitudes of the fundamental states play a crucial role. This is obviously due to the nonlinear effects which are proportional to the amplitudes of the initial states. It is possible to change the topological charge at a frequency only by changing the amplitude of the fundamental states. Controlling only the amplitudes allows one to reach three different states at a given frequency: 0 (for low amplitude), rm_1+sm_2, or pm_1+qm_2. Moreover, this change of topological charge will occur only for degenerate states and not the others.

Either optical or acoustical vortices were known to provide a mean to code digitized information in the wave front. In the nonlinear regime this information could be stored in the medium and reread or give rise to multiples of the topological charge through harmonics generation. We

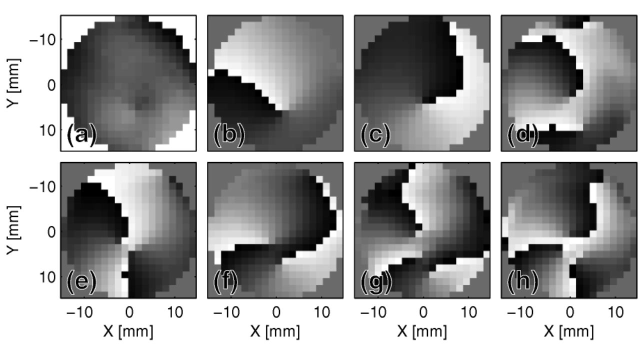

FIG. 3. Measured phase patterns for the first eight linear combinations of two noncommensurable primary frequencies ($f_1/f_2=29/50$). The displayed states are (a) $\mathbf{F}|f_2-f_1\rangle=0.525$ MHz, $\mathbf{M}|f_2-f_1\rangle=0$, (b) $\mathbf{F}|f_1\rangle=0.725$ MHz, $\mathbf{M}|f_1\rangle=1$ (initial state 1), (c) $\mathbf{F}|f_2\rangle=1.25$ MHz $\mathbf{M}|f_2\rangle=1$ (initial state 2), (d) $\mathbf{F}|2f_1\rangle=1.45$ MHz, $\mathbf{M}|2f_1\rangle=2$, (e) $\mathbf{F}|2f_2\rangle=2.5$ MHz, $\mathbf{M}|2f_2\rangle=2$, (f) $\mathbf{F}|f_1+f_2\rangle=1.975$ MHz, $\mathbf{M}|f_1+f_2\rangle=2$, (g) $\mathbf{F}|2f_1+f_2\rangle=2.7$ MHz, $\mathbf{M}|2f_1+f_2\rangle=3$, (h) $\mathbf{F}|f_1+2f_2\rangle=3.225$ MHz, $\mathbf{M}|f_1+2f_2\rangle=3$.

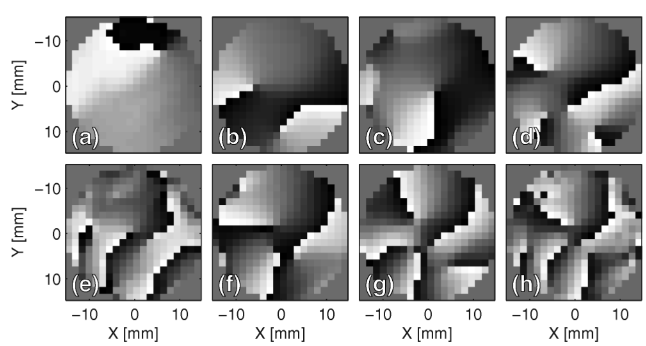

FIG. 4. Measured phase patterns for the first eight linear combinations of two noncommensurable primary frequencies ($f_1/f_2=19/33$). The displayed states are (a) $\mathbf{F}|f_2-f_1\rangle=0.7$ MHz, $\mathbf{M}|f_2-f_1\rangle=-1$, (b) $\mathbf{F}|f_1\rangle=0.95$ MHz, $\mathbf{M}|f_1\rangle=2$ (initial state 1), (c) $\mathbf{F}|f_2\rangle=1.65$ MHz $\mathbf{M}|f_2\rangle=1$ (initial state 2), (d) $\mathbf{F}|2f_1\rangle=1.9$ MHz, $\mathbf{M}|2f_1\rangle=4$, (e) $\mathbf{F}|2f_2\rangle=3.3$ MHz, $\mathbf{M}|2f_2\rangle=2$, (f) $\mathbf{F}|f_1+f_2\rangle=2.6$ MHz, $\mathbf{M}|f_1+f_2\rangle=3$, (g) $\mathbf{F}|2f_1+f_2\rangle=3.55$ MHz, $\mathbf{M}|2f_1+f_2\rangle=5$, (h) $\mathbf{F}|f_1+2f_2\rangle=4.25$ MHz, $\mathbf{M}|f_1+2f_2\rangle=4$.

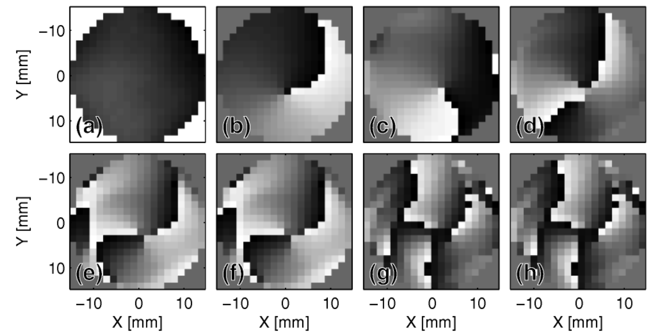

FIG. 5. Measured phase patterns for eight linear combinations of two commensurable primary frequencies ($f_1/f_2 = 2/3$). The displayed states are (a) $\mathbf{F}|f_2 - f_1\rangle = 0.5$ MHz, $\mathbf{M}|f_2 - f_1\rangle = 0$, (b) $\mathbf{F}|f_1\rangle = 1$ MHz, $\mathbf{M}|f_1\rangle = 1$ (initial state 1), (c) $\mathbf{F}|f_2\rangle = 1.5$ MHz, $\mathbf{M}|f_2\rangle = 1$ (initial state 2), (d) $\mathbf{F}|f_1 + f_2\rangle = 2.5$ MHz, $\mathbf{M}|f_1 + f_2\rangle = 2$, (e) $\mathbf{F}|2f_2\rangle = 3$ MHz, $\mathbf{M}|2f_2\rangle = 2$, (f) $\mathbf{F}|3f_1\rangle = 3$ MHz, $\mathbf{M}|3f_1\rangle = 3$, (g) $\mathbf{F}|3f_2\rangle = 4.5$ MHz, $\mathbf{M}|3f_2\rangle = 3$, (h) $\mathbf{F}|3f_1 + f_2\rangle = 4.5$ MHz, $\mathbf{M}|3f_1 + f_2\rangle = 4$.

have shown here that sums and subtractions can be performed in the nonlinear parametric interaction of two primary waves shifted in frequency. This new facility leads to different properties whether the frequencies of the two primary waves are commensurable or not. In the last case, the operation is unique for any nonlinear combination. Hence, in order to change the result one has to change the topological charge stored in the primary waves. This can be done dynamically if a bank of signals corresponding to different topological charges is computed and stored either in a computer or in the programmable electronic memories that drive the transducers. The same procedure could be carried out in optics by addressing and reconfiguring a SLM. On the contrary, in the first case, there exist frequencies resulting from more than one nonlinear combination so that the topological charge state is degenerate. Energetic thresholds delimiting different states appear. This degeneracy may be an interesting feature of this nonlinear process since a simple modulation in amplitude of the primary waves would lead to a shift between the different states or numbers. This effect is particularly appealing as the spatial pattern of the phase can be driven only with an amplitude parameter.

*marchi@lmm.jussieu.fr

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