Rainbow and Blueshift Effect of a Dispersive Spherical Invisibility Cloak Impinged On by a Nonmonochromatic Plane Wave

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(Received 17 April 2008; published 8 August 2008)

We demonstrate some interesting phenomena associated with a nonmonochromatic plane wave passing through a spherical invisibility cloak whose radial permittivity and permeability are of Drude and Lorentz types. We observe that the frequency center of a quasimonochromatic incident wave will suffer a blueshift in the forward scattering direction. Different frequency components have different depths of penetration, causing a rainbowlike effect within the cloak. The concept of group velocity at the inner boundary of the cloak needs to be revisited. Extremely low scattering can still be achieved within a narrow band.

DOI: 10.1103/PhysRevLett.101.063902 PACS numbers: 41.20.Jb, 42.25.Fx

Invisibility cloaking has recently received much attention in the literature [1–13]. Based on form-invariant coordinate transformations, a cloak of invisibility was proposed which can perfectly conceal arbitrary objects from detection [1,2]. Because of the fact that the physical realization of such a kind of cloak, i.e., metamaterials, must be dispersive [2,12,13], the ideal cloak can only work at a single frequency. In this Letter, the fundamental problem of how a nonmonochromatic electromagnetic wave passes through a dispersive spherical invisibility cloak is addressed, and some interesting phenomena are revealed.

In an ideal spherical cloak, the radial constitutive parameters ϵ_r and μ_r are required to vanish at the inner boundary [2] at the single frequency which can be named as the "cloaking frequency." Since any physical wave has a nonzero bandwidth, the transition of ϵ_r and μ_r from positive to negative will be formed within the cloak at frequencies entirely below or above the cloaking frequency because of dispersion. Similar positive-to-negative transition of constitutive parameters has been shown to cause some peculiar phenomena such as negative refraction [14] and superlens [15]. Resonances caused by surface polaritons between positive and negative index media have also been shown to produce strong anomalous scattering [16]. Therefore, it is necessary to analyze the influence of this transition of constitutive parameters on the performance of the cloak.

Based on a strict and efficient analytic model of a 3D dispersive and lossy spherical invisibility cloak, we show that the frequency center of a quasimonochromatic wave which is originally at the cloaking frequency will suffer a blueshift in the forward direction after passing this dispersive cloak. The singularity of wave equation inside the cloak will form impenetrable walls for electromagnetic waves. A rainbowlike effect will be formed since fields of different frequencies will penetrate into different depths inside the cloak. Meanwhile, the concept of group velocity at the inner boundary of the cloak is selectively valid only for those frequencies above the cloaking frequency. Extremely low RCS (radar cross section) can still be achieved for a narrow bandwidth. All of these aspects provide us with new insights into the cloaking phenomena in practice.

The configuration of a 3D spherical cloak with inner radius R_1 and outer radius R_2 follows that in Ref. [7]. The cloak shell within $R_1 < r < R_2$ is a radially uniaxial and inhomogeneous medium with permittivity tensor $\bar{\bar{\epsilon}} = \epsilon_r(r)\hat{r}\hat{r} + \epsilon_t\hat{\theta}\hat{\theta} + \epsilon_t\hat{\phi}\hat{\phi}$ and permeability tensor $\bar{\bar{\mu}} = \mu_r(r)\hat{r}\hat{r} + \mu_t\hat{\theta}\hat{\theta} + \mu_t\hat{\phi}\hat{\phi}$. An E_x polarized plane wave $\bar{E}_i = \hat{x}e^{ik_0 z}$ is incident upon the cloak. The electromagnetic wave in the cloak shell can be decomposed into TM and TE modes with respect to $\hat{r}$ [17], corresponding to scalar potentials $\Psi_{\rm TM}$ and $\Psi_{\rm TE}$ [7,11]. Since TM and TE modes have similar derivations, we focus on the former. The r dependent function $f(r)$ of $\Psi_{\rm TM}$ satisfies Eq. (1) (i.e., Eq. (4) in Ref. [7]), where $k_t^2 = \omega^2\epsilon_t\mu_t$.

$$\left\{\frac{\partial^2}{\partial r^2} + \left[k_t^2 - (\epsilon_t/\epsilon_r)\frac{n(n+1)}{r^2}\right]\right\}f(r) = 0. \qquad (1)$$

By utilizing the relation between constitutive parameters specified in [2] at the cloaking frequency, i.e. $\epsilon_r = \epsilon_t\frac{(r-R_1)^2}{r^2}$, $f(r)$ can be converted to the Riccati-Bessel function [7]. However, ϵ_t/ϵ_r is a function of frequency and position, which makes this specified relation no longer hold at a deviated frequency.

In order to solve this difficulty, we shall revisit Eq. (1). Since ϵ_r does not vary much with variation of r, we can divide the cloak shell in $R_1 < r < R_2$ into a lot of thin layers, which is similar to the recipe applied in practice [10]. Then both ϵ_r and ϵ_t in each layer of $R^{(j)} < r < R^{(j+1)}$ can be treated as constants, such that Eq. (1) has solutions

$$f(r) = a_{jn}\psi_\nu(k_t r) + a_{jn}R_{jn}^{\rm TM}\zeta_\nu(k_t r), \qquad j = 1, 2, \ldots, N \qquad (2)$$

where ψ_ν and ζ_ν are Riccati-Bessel functions of the first and the third kind, respectively, with a complex order

$\nu = \sqrt{\frac{\epsilon_t}{\epsilon_r} n(n+1) + \frac{1}{4}} - \frac{1}{2}$, and R_{jn}^{TM} is defined as the general reflection coefficient of nth order in the jth layer. The field solution in each layer can then be expressed with different coefficients a_{jn} and R_{jn}^{TM} as unknowns. By matching the boundary conditions between adjacent layers, all the coefficients can be solved. The coefficients for TE waves can be obtained similarly. In other words, the problem of solving the field solution in the whole space has been converted to solving a set of simultaneous linear equations which can be done in a straightforward manner.

To validate this algorithm, we study the dependence of normalized RCS (radar cross section normalized to πR_2^2) on the number of layers. The parameters at each layer's center are set to match those proposed in Ref. [2] and $R_2 = 2R_1 = 1.5\lambda_0$ at the cloaking frequency of 10 GHz, where $\lambda_0 = 3$ cm. The concealed region $r < R_1$ is specified to be PEC (perfect electric conductor). It can be found that as the number of layers increases, the RCS drops rapidly. When the number of layers reaches 100, the normalized RCS reaches 10^{-7}, which is extremely close to "perfect invisibility."

Since each layer's parameters can be specified arbitrarily, we are able to deal with more complicated situations including cases with anisotropic loss and dispersion. Since ϵ_t and μ_t are larger than 1 and do not vary within the whole cloak shell, they can be treated as constants over the frequency band of interest. The radial constitutive parameters ϵ_r and μ_r can be thought of as being achieved by embedding radially uniaxial metamaterials in a background with ϵ_t and μ_t. Subsequently, Drude model [18] and Lorentz model [19] are applied to ϵ_r and μ_r, respectively, as follows:

$$\epsilon_r = \epsilon_t\left(1 - \frac{f_p^2}{f(f + i\gamma_1)}\right), \tag{3}$$

$$\mu_r = \mu_t\left(1 - \frac{F}{1 + i\gamma_2/f - f_0^2/f^2}\right). \tag{4}$$

For simplicity, we set $\gamma_1 = \gamma_2 = \gamma$ and $F = 0.78$. Forcing the real parts of parameters at the center of each layer to match the relation proposed in Ref. [2] at the cloaking frequency, the corresponding f_p and f_0 for each layer can be calculated as well as subsequent ϵ_r and μ_r at other frequencies.

Now let us first consider the case where $\gamma \neq 0$. Then ϵ_r in Eq. (1) is always nonzero. Figure 1 shows the RCS spectrum of a dispersive cloak with different losses where the number of layers is set as $N = 100$. By decreasing γ, the RCS curve is convergent. It can be seen that though invisibility is sensitive to frequency deviation, extremely low RCS can still be obtained within a small finite bandwidth around the cloaking frequency. This result excludes the possibility of some kind of large anomalous scattering [16] in the vicinity of the cloaking frequency. Moreover, the working bandwidth of the cloak depends on the sensitivity of the detector outside. For example, if we set 0.04 as the upper limit of the undetectable normalized RCS, then the working bandwidth of the cloak is about 100 MHz around the cloaking frequency of 10 GHz. An arbitrary nonzero RCS limit can always be satisfied by narrowing the bandwidth.

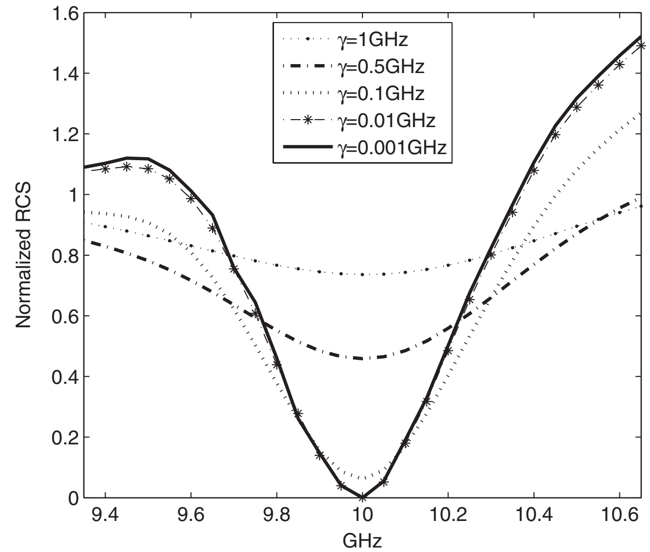

FIG. 1. Dependence of RCS (normalized to πR_2^2) on different frequencies and losses. $R_2 = 2R_1 = 1.5\lambda_0$. $\lambda_0 = 3$ cm. In the concealed region $r < R_1$ is PEC.

Next let us consider the other case where $\gamma = 0$. What is different in this case is that, if the frequency is below the cloaking frequency, then the normal dispersion requires that ϵ_r (or μ_r if TE waves are considered) close to the inner boundary be negative and Eq. (1) has a singularity at the position where ϵ_r is zero. This singularity will form an impenetrable wall for TM waves as we will demonstrate in the following. Similarly, another singularity caused by μ_r of zero value will form a wall for TE waves. Figure 2(a) shows the distribution of E_x in xz plane when a x-polarized incident plane wave ($\bar{E}_i = \hat{x}e^{ik_iz}$) is passing through the cloak along the z direction. For the sake of illustration, we choose the frequency deviation to be -532 MHz and $\gamma = 0.0001$ GHz. Since most contribution of E_x field in xz plane except near the z axis comes from TM waves, it can be seen that the TM field is expelled by a very thin "wall" within the shell which little TM field can penetrate. The position of this wall coincides with the position where ϵ_r is zero. In the yz plane which is not shown in this Letter, we can see a similar wall for H_y field except with a different location because μ_r has different dispersion and thus different location of zero value. The influence of loss on the field distribution is shown in Fig. 2(b) where the amplitude of E_x field along the direction ($\theta = 2\pi/3$, $\phi = 0$) where TM field is dominant is plotted. It can be seen that with decreasing γ, the field outside of the wall ($r > R_w^{\mathrm{TM}}$) is almost unchanged and the field inside of the wall ($r < R_w^{\mathrm{TM}}$) is decreasing fast while the wall at $r = R_w^{\mathrm{TM}}$ becomes thinner and sharper. Further decreasing of γ requires increasing the number of layers N which is not shown in

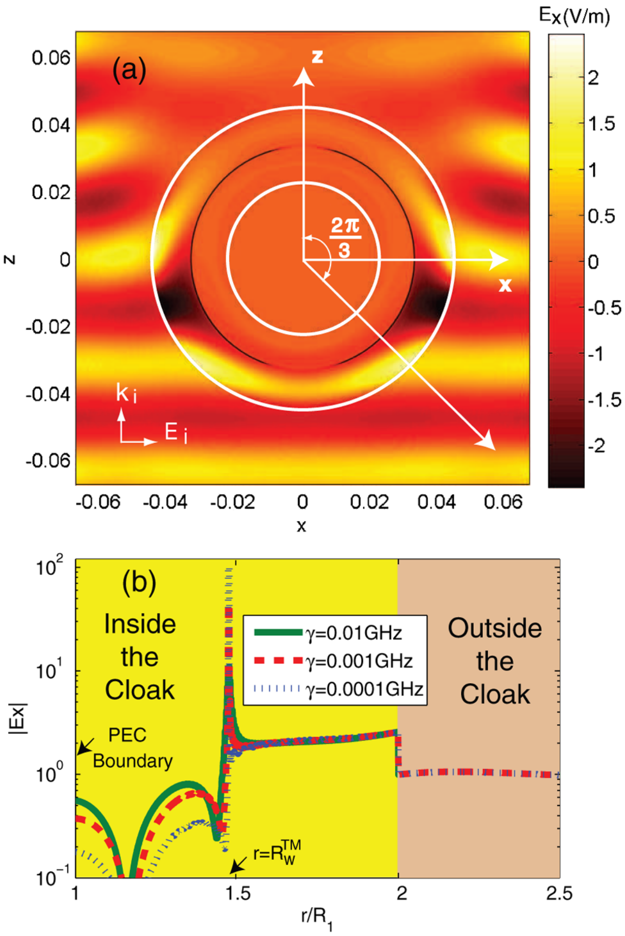

FIG. 2 (color online). Field distribution of (a) E_x in xz plane and (b) $|E_x|$ along the direction ($\theta = 2\pi/3$, $\phi = 0$) when a x-polarized incident plane wave with frequency 9.468 GHz is passing through the cloak along z direction. The size of the cloak is the same as Fig. 1. The number of layers $N = 300$.

Fig. 2. In the limit of $\gamma \rightarrow 0$, the field at $r = R_w^{\mathrm{TM}}$ (or $r = R_w^{\mathrm{TE}}$ for TE waves) will diverge and all TM fields will be blocked for $r > R_w^{\mathrm{TM}}$ while for TE fields, for $r > R_w^{\mathrm{TE}}$. Similar wall effects have been found to prevent waves excited by an active device inside the concealed region from going out at the cloaking frequency [11]. But here this effect is induced by the incoming wave directly when the frequency is deviated from the cloaking frequency. A more rigorous argument can be made as follows. Following the derivation in Ref. [11], suppose a layer within the cloak shell has ϵ_r and thickness d, then it can be found that in the limit of $\epsilon_r \rightarrow 0$, the TM field inside this layer as well as all the space it encloses decreases to zero while the TM field on the surface of the layer diverges, no matter how small d is. Since this wall for TM waves requires the tangential H field to vanish [11], it can be treated as a PMC (perfect magnetic conductor) wall. Similarly, another wall expelling TE waves is a PEC wall.

This peculiar wall effect will form a rainbowlike field distribution inside the cloak since different frequency components penetrate into different depths from $r = R_2$ to $r = R_1$, as shown in Fig. 3 where six frequencies are considered. Such a frequency selection challenges a basic concept in optics, group velocity. The group velocity at the inner boundary of the cloak has been calculated from the point of view of geometrical optics to study causality of the cloak [12,13]. However, what really happens at the inner boundary should be based on the field solution directly. As we all know, the group velocity at the cloaking frequency of f_c represents the speed of the envelop formed by two close frequency components $f_c - \delta f$ and $f_c + \delta f$. For the wave with frequency $f_c + \delta f$, it is able to reach the inner boundary $r = R_1$. But for the wave with frequency $f_c - \delta f$, as we have shown, will be stopped somewhere in-between by the PMC and PEC walls and thus never reach the inner boundary $r = R_1$. Therefore the group velocity at $r = R_1$ is meaningful only for the frequencies above the cloaking frequency, i.e., $f > f_c$. As a result, the group velocity at $r = R_1$ at the cloaking frequency is meaningless.

The different responses of cloak to different frequencies can lead to another interesting phenomenon. At the cloaking frequency f_c, the strictly monochromatic wave will pass through the cloak without any distortion [2,4,7]. So an observer looking at an object emitting or reflecting this monochromatic wave behind the cloak will see exactly the same object as if there is no block in front of it. But is this true for a more physical quasimonochromatic wave possessing a narrow band? For the wave with frequency slightly deviated above at $f_c + \delta f$, the most part of the cloak shell does not change much except the part close to the inner boundary $r = R_1$. Thus only a narrow spectrum of wave can reach the PEC core within $r < R_1$. From the view of the observer outside, the scattering looks as if it is from a very small PEC particle; i.e., Rayleigh scattering occurs in this case [12]. For a very small PEC particle with $kR \ll 1$, the first order scattering coefficients R_{01}^{TM} and R_{01}^{TE} become dominant [20], where $R_{01}^{\mathrm{TM}} = -\psi'(kR)/\zeta'(kR) \approx 2i/3(kR)^3$ and $R_{01}^{\mathrm{TE}} = -\psi(kR)/\zeta(kR) \approx -i/3(kR)^3$. The phase of scattering wave in the forward direction ($\theta = 0$) depends on the phase of $-i\sum(R_{0n}^{\mathrm{TM}} + R_{0n}^{\mathrm{TE}})$ [21] which in this case is $-i(R_{01}^{\mathrm{TM}} + R_{01}^{\mathrm{TE}}) \approx +1/3(kR)^3$. The positive

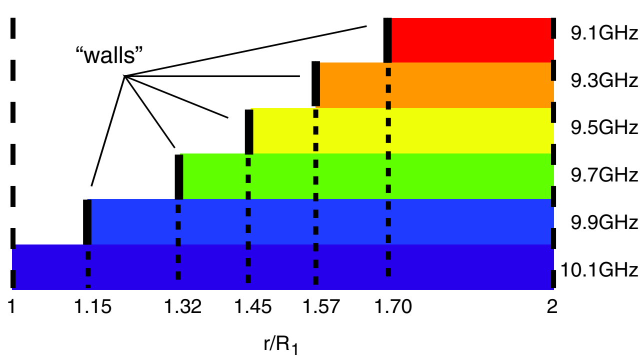

FIG. 3 (color online). Fields of different frequencies have different depths of penetration within the cloak. Here only TM fields are considered. All other parameters follow Fig. 2.

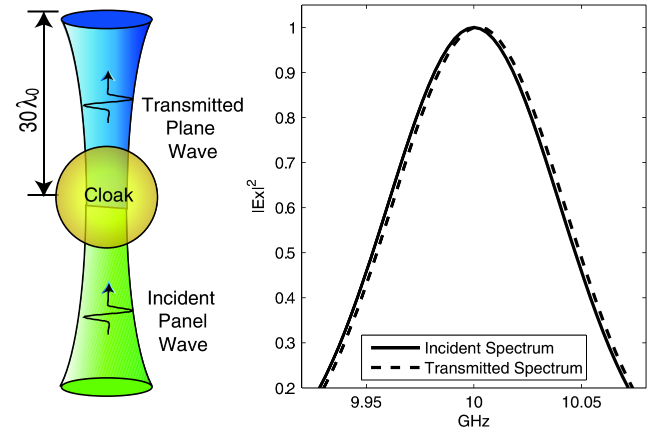

FIG. 4 (color online). The frequency center of a quasimonochromatic incident plane wave is blueshifted in the forward direction after passing through the cloak. The dimension of the cloak follows Fig. 1. The transmitted wave is measured at the location $30\lambda_0$ apart from the cloak.

sign indicates that the forward scattering field will constructively interfere with the original incident field. Thus the wave after passing the cloak will be reinforced. But for the incident wave with frequency slightly deviated below at $f_c - \delta f$, the observer outside will regard the scattering of TM waves as if it is from a PMC particle and the scattering of TE waves as if it is from a PEC particle because of the PMC and PEC walls close to the inner boundary. Thus in this case $R_{01}^{\rm TM} = R_{01}^{\rm TE} = -\psi(kR)/\zeta(kR) \approx -i/3(kR)^3$, which gives $-i(R_{01}^{\rm TM} + R_{01}^{\rm TE})$, a negative value. Thus the field after the wave passes the cloak will be decreased. After noting these differences, we can study the behavior of this quasimonochromatic wave passing through the cloak. Since the frequency components $f > f_c$ are reinforced while the other frequency components $f < f_c$ are weakened, the frequency center of this incident wave will be blueshifted after passing the cloak in the forward direction, i.e., shifted toward a higher frequency. Thus the observer looking at the object emitting or reflecting this quasimonochromatic wave behind the cloak will see a blueshifted wave. Figure 4 plots this shift effect for a quasimonochromatic incident plane wave with Gaussian distribution in frequency spectrum with mean of 10 GHz and variation of 50^2 MHz^2. The transmitted wave is measured at the location $30\lambda_0$ apart from the cloak. It should be noted that this blueshift is caused by the shift of frequency center of the narrow band wave, which does not mean any change of frequency itself. Furthermore, this blueshift effect does not depend on the size of the cloak, but the normal dispersions of radial constitutive parameters. For anomalous dispersions which rarely occur, a redshift effect will arise.

In conclusion, the field solution for a 3D dispersive spherical invisibility cloak in response to a nonmonochromatic electromagnetic wave is established. The singularity of wave equation inside the cloak will form impenetrable walls for electromagnetic waves. A rainbowlike field distribution inside the cloak will be formed since different frequencies have different depths of penetration. The frequency center of a quasimonochromatic wave with narrow band will be blueshifted in the forward direction after passing through the cloak. The group velocity at the inner boundary of the cloak is meaningful only when the frequency is above the cloaking frequency. Extremely low RCS can still be achieved within a narrow band around the cloaking frequency.

This work is sponsored by the ONR under Contract No. N00014-01-1-0713, the Department of the Air Force under Air Force Contract No. F19628-00-C-0002, and the Chinese NSF under Grant No. 60531020.

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