Singularity Problem with $f(R)$ Models for Dark Energy

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In this Letter, I point out that there is a curvature singularity problem appearing on the nonlinear level that generally plagues $f(R)$ models that modify Einstein gravity in the infrared. It is caused by the fact that for the effective scalar degree of freedom, the curvature singularity is at a finite field value and energy level, and can be easily accessed by the field dynamics in the presence of matter. This problem is invisible in a linearized analysis, except for the telltale growing oscillatory modes it causes. In view of this, the viability of many $f(R)$ models in the current literature will have to be reevaluated.

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What is causing the observed accelerated expansion of the Universe today is one of the biggest open questions in modern cosmology. Trying to explain it by modifying the theory of gravity rather than by introducing a mysterious dark energy component has been a popular pursuit as of late. Unfortunately, it is proving to be a rather difficult thing to do consistently while avoiding a variety of stringent observational tests of gravity we have at our disposal. A class of such models that received much attention recently is the one that modifies the Einstein-Hilbert gravitational action by replacing Ricci curvature scalar by an arbitrary function of the curvature

$$S = \int \left\{ \frac{f(R)}{16\pi G} + \mathcal{L}_m \right\} \sqrt{-g} d^4x. \tag{1}$$

Introduced in cosmological context for the case which modifies gravity in the high energy limit in a seminal paper by Starobinsky [1] and studied in [2–4], this model has later been adopted for infrared modifications of gravity as well [5,6]. For the latter application, it turned out to be not without problems. Certain constraints have to be imposed on function $f(R)$ for the model to be linearly stable [7] and cosmologically viable [8–10]. The first attempts failed these constraints right away, but since then, models that evade them have been found (for example, see [11–13] and references therein) and enough trust has been placed in their viability to study cosmological structure formation in detail [14,15].

In this Letter, I point out a serious curvature singularity problem that affects many, if not all, infrared-modified $f(R)$ models. Being nonlinear in nature, it has escaped scrutiny so far.

As it is well known, a new scalar degree of freedom appears in $f(R)$ gravity that is not there in Einstein theory (sometimes dubbed the *scalaron*). Conformal transformation of the metric can be employed to make it explicit in the action [3,4]. In this Letter, I will avoid doing that to keep the usual matter coupling to the metric, and work with the action (1) directly. Variation with respect to metric yields gravitational equations of motion

$$f' R_{\mu\nu} - f'_{;\mu\nu} + \left(\Box f' - \frac{1}{2} f \right) g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2}$$

where prime ($\prime$) denotes the derivative of the function f with respect to its argument R, and $\Box$ is the usual notation for covariant D'Alembert operator $\Box \equiv \nabla_\alpha \nabla^\alpha$. The equation of motion for a new scalar degree of freedom is given by the trace of Eq. (2)

$$\Box f' = \frac{1}{3}(2f - f'R) + \frac{8\pi G}{3} T. \tag{3}$$

Identifying the scalar degree of freedom explicitly by a variable redefinition $\phi \equiv f' - 1$, the above equation is cast in the form of equation of motion of a canonical dimensionless scalar field ϕ with a potential V and a force term $\mathcal{F}$

$$\Box \phi = V'(\phi) - \mathcal{F}. \tag{4}$$

The effective scalar field potential $V(\phi)$ is determined by

$$V'(\phi) \equiv \frac{dV}{d\phi} = \frac{1}{3}(2f - f'R) \tag{5}$$

expressed in terms of the scalar variable ϕ. In practice, given $f(R)$, it is usually difficult to invert the definition of the scalar degree of freedom explicitly, so it might be more convenient to determine effective potential V in a parametric form instead. By integrating

$$\frac{dV}{dR} \equiv \frac{dV}{d\phi} \frac{d\phi}{dR} = \frac{1}{3}(2f - f'R) f'', \tag{6}$$

potential $V(\phi)$ is then given by a pair of functions $\{\phi(R), V(R)\}$. The force term $\mathcal{F}$ that drives the scalar field ϕ is a trace of the stress-energy tensor T, which for perfect fluid is simply $\mathcal{F} = (8\pi G/3)(\rho - 3p)$.

Let us consider a homogeneous cosmological model in $f(R)$ gravity, with the usual complement of matter fields. Expansion of the Universe is described by a flat Friedman-Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t) d\mathbf{x}^2, \tag{7}$$

and the scalar gravitational degree of freedom ϕ obeys a usual scalar field equation, albeit with a force term on the right-hand side

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = \mathcal{F}. \qquad (8)$$

The analog of Friedmann equation in $f(R)$ cosmology is not so transparent. Let us consider the tt component of gravitational equations of motion (2). For metric (7), it is

$$3H(f')^{\cdot} - 3\frac{\ddot{a}}{a}f' + \frac{1}{2}f = 8\pi G\rho. \qquad (9)$$

Note that unlike the usual Friedmann equation, higher derivatives of scale factor a appear. Second derivative $\ddot{a}$ is written out explicitly, and a third derivative is hiding in a time derivative of f' term, which itself contains Ricci curvature, and hence $\ddot{a}$. Seeing a second derivative of the scale factor, one might be tempted to treat the above Eq. (9) as a dynamical evolution equation for the scale factor. Doing so, however, is not a very good idea. For small deviations from Einstein gravity, the coefficient in front of $\ddot{a}$ goes degenerate, and Eq. (9) does not have a good limit determining $\ddot{a}$ (which is not all that surprising, considering that the Friedmann equation in Einstein gravity does not constrain $\ddot{a}$ directly). To get a proper limit, let us instead get rid of $\ddot{a}$ in favor of the curvature scalar $R = 6(\ddot{a}/a + \dot{a}^2/a^2)$. After that is done, Eq. (9) becomes

$$H^2 + (\ln f')^{\cdot} H + \frac{1}{6}\frac{f - f'R}{f'} = \frac{8\pi G}{3f'}\rho, \qquad (10)$$

and its role as a constraint equation is revealed. In the limit of Einstein gravity, $f' \to 1$, and so the last two terms on the left-hand side disappear, and one is left with the usual Friedmann equation. In the general case, the extra terms are functions of scalar degree of freedom ϕ and its first time derivative. No higher derivatives appear in this equation anymore.

Thus, the following simple picture of dynamics in the $f(R)$ cosmology emerges. Above the infrared modification scale R_0, the expansion rate of the Universe is set primarily by the matter density, just like in the usual cosmology, with small corrections. Only once the local curvature drops below R_0, the expansion rate starts feeling the effect of gravity modification. The spacetime curvature, on the other hand, is controlled by the scalar degree of freedom ϕ which gravity acquires. It obeys the usual scalar field Eq. (8) with potential $V(\phi)$, the shape of which is directly determined by function $f(R)$, and a driving term from the trace of matter stress-energy tensor.

But here is the problem: it turns out that precisely those functions $f(R)$ that lead to Einstein-like gravity action in the large curvature regime, yield a potential V with an unprotected curvature singularity.

As a case in point, consider Starobinsky's disappearing cosmological constant model [12], which has been very carefully constructed and avoids all known linear instabilities. It is described by

$$f(R) = R + \lambda[(1 + R^2)^{-n} - 1], \qquad (11)$$

where I have taken a liberty to absorb the crossover curvature scale R_0 into rescaling of coordinates (which become dimensionless and are measured in length units corresponding to R_0). For definiteness, let us take $n = 1$. The scalar degree of freedom in this model is given by

$$\phi = -\frac{2\lambda R}{(1 + R^2)^2} \qquad (12)$$

in terms of curvature, so large curvature limit $R \to \pm\infty$ corresponds to $\phi = 0$. Flat spacetime with $R = 0$ also corresponds to $\phi = 0$, which gives us a hint that the potential is going to be a multivalued function. The potential can be evaluated by integrating (6); up to an arbitrary constant, it is

$$V = \frac{\lambda^2 R(3 + 11R^2 + 21R^4 - 3R^6)}{24(1 + R^2)^4} - \frac{\lambda R^2(1 + R^2 - R^4 - R^6)}{3(1 + R^2)^4} - \frac{\lambda^2}{8}\arctan R. \qquad (13)$$

The effective scalar potential is plotted in Fig. 1 for $\lambda = 2$, and is indeed multivalued. Let us walk through the interesting locations on this plot. Point A is a positive curvature singularity $R = +\infty$. Point B is the stable de Sitter minimum in this model, and point C is the unstable de Sitter maximum; their curvatures depend on λ. Point E corresponds to a flat spacetime, which although a solution in this model, is unstable. Points D and F are critical points with $f'' = 0$ that occur at $R = \pm 1/\sqrt{3}$; potential branches there. Finally, point G is a negative curvature singularity $R = -\infty$. Only the small part of this potential is actually relevant for cosmological evolution from initial singularity to today, and it lies in the arc AB, shaded blue in Fig. 1.

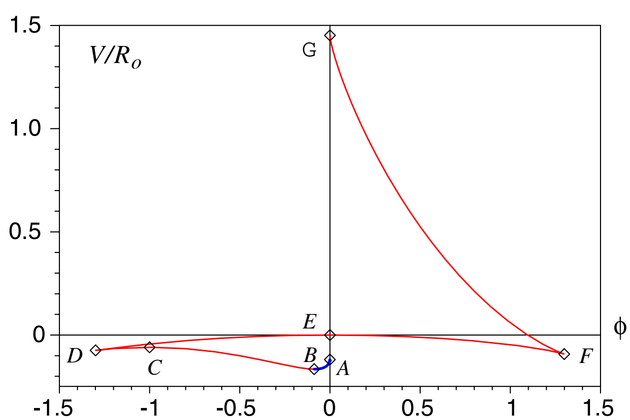

FIG. 1 (color online). Effective potential of a scalar degree of freedom in $f(R)$ gravity model (11) with $\lambda = 2$ and $n = 1$. Diamonds mark the location of critical points. The part relevant to cosmological evolution is emphasized by the thick line.

The most striking feature of the potential in Fig. 1, and the core of the problem for infrared-modified $f(R)$ models is that curvature singularity at point A is a finite distance away both in field and energy values from the place we are supposed to live in. Scalar degree of freedom ϕ directly feels the matter distribution through the force term; for equation of state $w < 1/3$, the force is directed to the right and drives the field ϕ up the wall toward point A and infinite curvature. Characteristic scale of the potential V is the crossover curvature scale R_0, and hence of the same order of magnitude as a present day cosmological constant, which is exceedingly low compared to matter densities we encounter every day. Given the scales involved, it appears to be quite easy to overdrive the scalar degree of freedom and make it "jump out" of the potential well by doing simple manipulations with normal matter (say a pile of dust), which would cause catastrophic curvature singularity. Needless to say, if this were to happen, it would not make for a desirable (or even viable) model. Similarly, but less dramatically, matter with sufficiently stiff equation of state can destabilize the model by driving the field to the left past the unstable point C.

The presence of the curvature singularity a finite distance away is extremely disturbing by itself, but let us examine more carefully if it is reached by physically reasonable solutions. Inside a constant density matter distribution, one can think of a (constant) force term $\mathcal{F}$ as coming from a linear field potential $\mathcal{F}(\phi_* - \phi)$ instead, and introduce a new "in matter" effective potential

$$U(\phi) = V(\phi) + \mathcal{F}(\phi_* - \phi), \tag{14}$$

where ϕ_* denotes the asymptotic de Sitter vacuum field. The comparison between the two potentials V and U for $\mathcal{F} = R_0$ is shown in the Fig. 2. As you can see, addition of matter slopes the potential U, shifts the (stable) minimum to the right but makes it more shallow, and lowers the curvature singularity point X. The density needed to make the curvature singularity energetically accessible from vacuum B is given by the ratio of potential barrier $\delta V = V_A - V_B$ to the scalar field value distance from vacuum to singularity $\delta\phi = \phi_A - \phi_B = -\phi_*$. It is of the same order as the density of dark energy today, with a numerical factor which depends on the model. So for a vast majority of physical solutions with matter, the curvature singularity is energetically accessible from asymptotic de Sitter vacuum, and the potential minimum is so close to curvature singularity that it would be invisible if actually plotted to scale.

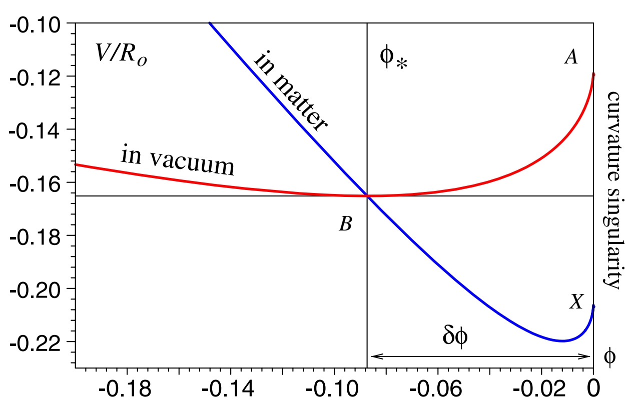

FIG. 2 (color online). Adding matter destabilizes the vacuum. Although effective potential inside constant density matter distribution still has a minimum, it is very shallow and cannot protect the field ϕ from reaching curvature singularity X, which becomes energetically accessible from asymptotic vacuum state B.

Energetical accessibility of the curvature singularity causes problems. For example, if one takes a cosmological solution approaching dark energy domination today and traces it back into the past, one is very likely to encounter a curvature singularity. This has been noticed numerically [16,17], but the underlying reasons for it and the extent of the damage were not fully realized. It is also most likely the cause of growing oscillatory curvature modes [12] which signal the breakdown of linear expansion due to closeness of potential minimum to curvature singularity. Although a more detailed analysis of the approach to singularity is in order, from Eq. (10), it appears that the singularity occurs at finite redshift, density and expansion rate and is driven by the divergence of the second derivative of the scale factor $\ddot{a}$ (which would make it rather weak, but a singularity nonetheless).

Although I focused on cosmology so far, perhaps a more deadly argument against having a curvature singularity at finite distance in the field space comes from considering a gravitational field of a static dense compact object (like a neutron star). Although the exact nonlinear solution of this problem is more complicated to analyze [18,19] and is beyond the scope of this Letter, I can give a very simple estimate if the problem occurs. As we have seen, the energetics of the scalar gravitational degree of freedom are by far dominated by the matter driving term. If we discard the contribution of nonlinear potential V (which is negligible everywhere except maybe very close to singularity at $\phi = 0$ for compact object), the equation for gravitational field of a static matter distributions becomes a simple Laplace equation

$$\Delta\phi = -\frac{8\pi}{3} G\rho. \tag{15}$$

Comparing this with an equation for Newtonian gravitational potential

$$\Delta\Phi = 4\pi G\rho, \tag{16}$$

we get a simple estimate $\phi = \phi_* - 2\Phi/3$ for the excitation of scalar gravitational degree of freedom ϕ in $f(R)$ gravity in terms of Newtonian potential well depth Φ of the compact object, where ϕ_* is the asymptotic value of ϕ at infinity, i.e., the minimum value $\phi_B = -\delta\phi$. But unlike Newtonian potential Φ, which has to diverge to cause singularity or reach $-1/2$ to form a horizon, gravitational degree of freedom ϕ needs only to change by a (small) amount $\delta\phi$ from its vacuum value to create a singularity. So unless an infrared-modified $f(R)$ model leads to a potential with curvature singularity separated from vacuum by at least $\delta\phi \gtrsim 1/3$, one would end up with a curvature singularity *without horizon* in a compact astrophysical object like a neutron star. This condition is rather easy to violate unless special care is taken in model-building. For example, for Starobinsky's model (11) with $n = 1$ and $\lambda = 2$ (as in Fig. 1) $\delta\phi \simeq 0.0874 \ll 1/3$, and is even smaller for larger values of λ, for which it decreases as $\delta\phi \sim (2\lambda)^{-2}$. Since in general one needs $f' > 0$ for graviton not to be a ghost, one would need $-1 < \phi_* \lesssim -1/3$ to avoid both problems, the prospects of achieving which without fine-tuning do not look good.

This curvature singularity problem is in no way unique to Starobinsky's disappearing cosmological constant model [12], which I have taken as an example simply because it is one of the most carefully constructed models so far. In fact, any infrared-modified $f(R)$ gravity model suffers from it. Let us consider arbitrary function $f(R)$, and require that it reduces to Einstein gravity for large curvature and has an analytic expansion

$$f(R) = R + \Lambda + \frac{1}{R^\alpha}\sum_{n=0}^{\infty}\frac{\mu_n}{R^n} \tag{17}$$

with a leading term μ_0/R^α (with $\alpha > 0$). Then, the leading terms for large R asymptotic behavior of scalar gravitational degree of freedom and potential (6) are

$$\phi \simeq -\frac{\alpha\mu_0}{R^{\alpha+1}}, \qquad V \simeq \text{const} - \frac{(\alpha+1)\mu_0}{3R^\alpha}. \tag{18}$$

The value of ϕ goes to zero in large curvature limit, and the potential V has power law dependence on ϕ

$$V(\phi) \simeq \text{const} - \frac{(\alpha+1)\mu_0}{3|\alpha\mu_0|^\gamma}|\phi|^\gamma, \qquad \gamma = \frac{\alpha}{\alpha+1}, \tag{19}$$

with exponent γ valued between zero and one. Thus, the values of both the field and the potential at curvature singularity are finite for a generic $f(R)$ infrared modification of gravity which recovers Einstein gravity perturbatively in the large curvature limit. This means the arguments I made above apply generically, and viability of many $f(R)$ models in current literature will have to be reevaluated. At the very least, the bound for compact objects will have to be satisfied for the model not to be ruled out immediately. But even if the estimate I made here looks safe, any infrared-modified $f(R)$ models should be scrutinized very closely for dangerous curvature singularities that could be present. In a sense, infrared modification of $f(R)$ gravity forces one to confront the question of ultraviolet completion of the theory.

Finally, let me comment on how this problem looks in equivalent scalar-tensor theory formulation [3,4]. Conformal transformation to an Einstein frame with metric $d\hat{s}^2 = f' ds^2$ turns the scalar degree of freedom into a canonically normalized scalar field ψ with potential

$$\psi = \sqrt{\frac{2}{3}}\ln f', \qquad W(\psi) = \frac{1}{2}e^{-(4\psi/\sqrt{6})}(Rf' - f). \tag{20}$$

The asymptotics of scalar degree of freedom in the Einstein frame are very similar to the above story: the field ψ goes to zero in large curvature limit, and the potential has the same unprotected power law asymptotic $W \simeq a - b|\psi|^\gamma$. But where did the singularity go? The answer is subtle: while the conformal factor itself appears to be regular ($f' \to 1$), its second derivatives are not (potential derivative blows up as $|\psi|^{\gamma-1}$ in equation of motion), which can cause a curvature singularity in the Jordan frame even if the Einstein frame metric was regular.

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