Coulomb Blockade and Superuniversality of the θ Angle

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Based on the Ambegaokar-Eckern-Schön approach to the Coulomb blockade, we develop a complete quantum theory of the single electron transistor. We identify a previously unrecognized physical observable in the problem that, unlike the usual average charge on the island, is robustly quantized for any *finite* value of the tunneling conductance as the temperature goes to absolute zero. This novel quantity is fundamentally related to the nonsymmetrized current noise of the system. Our results display all of the superuniversal topological features of the θ angle concept that previously arose in the theory of the quantum Hall effect.

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The Ambegaokar-Eckern-Schön (AES) model [1] is the simplest approach to the Coulomb blockade problem [2,3] that has attracted a considerable amount of interest over the years. The standard experimental setup is the single electron transistor (SET), which is a mesoscopic metallic island coupled to a gate and connected to two metallic reservoirs by means of tunneling contacts with a total conductance g. Even though the physical conditions of the AES model are limited and well known [4–6], the theory nevertheless displays richly complex and fundamentally new behavior, much of which has not been understood to date. To explain the observed tunneling phenomena with varying temperature T and gate voltage V_g, one usually considers an isolated island obtained by putting the tunneling conductance g equal to zero. The AES model then leads to the standard semiclassical or electrostatic picture of the Coulomb blockade, which says that at $T = 0$ the average charge (Q) on the island is *robustly* quantized except for very special values of the gate voltage $V_g^{(k)} = e(k + 1/2)/C_g$, with integer k and C_g denoting the gate capacitance. At these very special values, a *quantum phase transition* occurs where the average charge Q on the island changes from $Q = k$ to $Q = (k + 1)$ in units of e.

The experiments on the SET always involve *finite* values of the tunneling conductance g, however, and this dramatically complicates the semiclassical picture of the Coulomb blockade. Despite ample theoretical work on both the strong and the weak coupling sides of the problem, the matter still lacks basic physical clarity since the averaged charge Q is known to be *unquantized* for any finite value of g, no matter how small [7]. This raises fundamental questions about the exact meaning of the experiments and the physical quantities in which the Coulomb blockade is usually expressed.

In this Letter, we present a complete quantum theory of the SET that is motivated by the formal analogies that exist between the AES theory, on the one hand, and the theory of the quantum Hall effect [8], on the other. Each of these theories describe an interesting experimental realization of the topological issue of a θ vacuum that originally arose in QCD [9]. In each case, one deals with different physical phenomena and therefore different quantities of physical interest. What has remarkably emerged over the years is that the basic scaling behavior is always the same, independent of the specific application of the θ angle in which one is interested [8]. Within the Grassmannian $U(m + n)/U(m) \times U(n)$ nonlinear σ model, for example, one finds that quantum Hall physics is, in fact, a superuniversal topological feature of the theory for all values of m and n. It is therefore of interest to know whether superuniversality is retained in the AES theory where physical concepts such as the "Hall conductance" and "θ renormalization" have not been recognized.

In direct analogy with the theory of the quantum Hall effect, we develop, in the first part of this Letter, a quantum

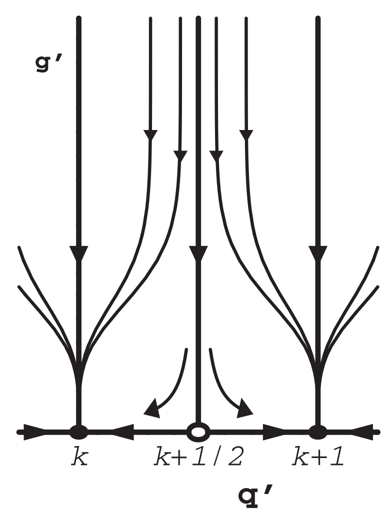

FIG. 1. Unified scaling diagram of the Coulomb blockade in terms of the SET conductance g' and the quasiparticle charge q'. The arrows indicate the scaling toward $T = 0$ (see text).

theory of *observable parameters* g', E_c', and q' for the AES model obtained by studying the *response* of the system to changes in the boundary conditions. Here g' is identified as the SET conductance and E_c' is the charging energy, whereas q' is a novel physical quantity that is fundamentally related to the current noise in the SET. q' is in all respects the same as the Hall conductance in the quantum Hall effect, and, unlike the averaged charge Q on the island, it is robustly quantized in the limit $T \to 0$. The crux of this Letter is the unifying scaling diagram of Fig. 1 indicating that g' and q' are the appropriate renormalization group parameters of the AES theory. These scaling results, which are the main objective of the remaining part of this Letter, provide the complete conceptual framework in which the various disconnected pieces of existing computational knowledge of the AES theory can, in general, be understood.

AES model.—The action involves a single Abelian phase $\phi(\tau)$ describing the potential fluctuations on the island $V(\tau) = i\dot{\phi}(\tau)$, with τ denoting the imaginary time [1]. The theory is defined by

$$Z = \int \mathcal{D}[\phi] e^{-\mathcal{S}[\phi]}, \qquad \mathcal{S}[\phi] = \mathcal{S}_d + \mathcal{S}_t + \mathcal{S}_c. \quad (1)$$

The action $\mathcal{S}_d$ describes the tunneling between the island and the reservoirs

$$\mathcal{S}_d[\phi] = \frac{g}{4}\int_0^\beta d\tau_1 d\tau_2 \alpha(\tau_{12}) e^{-i[\phi(\tau_1)-\phi(\tau_2)]}. \quad (2)$$

Here $\beta = 1/T$, $\tau_{12} = \tau_1 - \tau_2$, and $g = g_l + g_r$, where $g_{l,r}$ denotes the dimensionless bare tunneling conductance between the island and left or right reservoir. The kernel $\alpha(\tau)$ is usually expressed as $\alpha(\tau) = (T/\pi)\sum_n |\omega_n| e^{-i\omega_n \tau}$, with $\omega_n = 2\pi T n$. The part $\mathcal{S}_t$ describes the coupling between the island and the gate, and $\mathcal{S}_c$ is the effect of the Coulomb interaction between the electrons

$$\mathcal{S}_t[\phi] = -2\pi i q \mathcal{C}[\phi], \qquad \mathcal{S}_c[\phi] = \frac{1}{4E_c}\int_0^\beta d\tau \dot{\phi}^2. \quad (3)$$

Here $q = C_g V_g/e$ is the external charge, and $\mathcal{C}[\phi] = 1/(2\pi)\int_0^\beta d\tau \dot{\phi}$ is the winding number or *topological charge* of the ϕ field. For the system in equilibrium, the winding number is strictly an integer [5] which means that Eq. (3) is sensitive only to the *fractional* part $-\frac{1}{2} < q < \frac{1}{2}$ of the external charge q. The main effect of $\mathcal{S}_c$ in Eq. (3) is to provide a cutoff for large frequencies. Equation (2) has classical finite action solutions $\phi_0(\tau)$ with a nonzero winding number that are completely analogous to Yang-Mills instantons. The general expression for winding number W is given by [10,11]

$$e^{i\phi_0(\tau)} = e^{-i2\pi T\tau}\prod_{a=1}^{|W|}\frac{e^{i2\pi T\tau} - z_a}{e^{-i2\pi T\tau} - z_a^*}. \quad (4)$$

For instantons ($W > 0$) the complex parameters z_a are all inside the unit circle, and for anti-instantons ($W < 0$) they are outside. Considering $W = \pm 1$, which is of interest to us, one identifies $\arg z/2\pi T$ as the *position* (in time) of the single instanton, whereas $\lambda = (1 - |z|^2)\beta$ is the *scale size* or the duration of the potential pulse $i\dot{\phi}_0(\tau)$. The semiclassical expression for the thermodynamic potential $\Omega = -T\ln Z$ [12] can be written in the standard instanton form [8,9]

$$\Omega_{\text{inst}} = -gD\int_0^\beta \frac{d\lambda}{\lambda^2} e^{-[g(\lambda)/2]-\mathcal{O}(1/\lambda E_c(\lambda))}\cos 2\pi q. \quad (5)$$

Here $D = 2e^{-\gamma_E - 1}$, with $\gamma_E \approx 0.577$ denoting the Euler constant. By introducing a frequency scale $\nu_0 = gE_c/(\pi^2 D)$, then $g(\lambda)$ and $E_c(\lambda)$ are given by [13]

$$g(\lambda) = g - 2\ln\lambda\nu_0, \qquad E_c(\lambda) = E_c\left[1 - \frac{2}{g}\ln\lambda\nu_0\right]. \quad (6)$$

The logarithmic corrections are the same as those computed in ordinary perturbation theory in $1/g$. Based on Eq. (6) alone, one expects that the SET always scales from a good *conductor* at high T or short times $\lambda\nu_0 \ll 1$ to an *insulator* at low T or long times $\lambda\nu_0 \gg 1$.

Kubo formulas.—To obtain the low energy dynamics of the SET, we employ the background field $\tilde{\phi}(\tau) = \omega_n \tau$ that satisfies the classical equation of motion of Eq. (1). The background field action $\mathcal{S}'[\tilde{\phi}]$

$$e^{-\mathcal{S}'[\tilde{\phi}]} = Z^{-1}\int \mathcal{D}[\phi] e^{-\mathcal{S}[\tilde{\phi}+\phi]} \quad (7)$$

is properly defined in terms of a series expansion in powers of ω_n. To lowest orders we can write

$$\mathcal{S}'[\tilde{\phi}] = \beta\left(\frac{g'}{4\pi}|\omega_n| - iq'\omega_n + \frac{\omega_n^2}{4E_c'}\right), \quad (8)$$

which is the same as the classical action $\mathcal{S}[\tilde{\phi}]$ except that the *bare* parameters g, q, and E_c are replaced by *observable* ones. The observable theory is formally given in terms of Kubo-like expressions

$$\begin{aligned} g' &= 4\pi \operatorname{Im}\langle K(\eta)\rangle|_{\eta\to 0}, \\ q' &= q + \frac{i\langle\dot{\phi}\rangle}{2E_c} + \operatorname{Re}\langle K(\eta)\rangle\Big|_{\eta\to 0}, \\ \frac{1}{E_c'} &= \frac{1}{E_c}\left(1 + \int_0^\beta d\tau e^{i\eta\tau}\langle\dot{\phi}(\tau)K(\eta)\rangle\right)\Big|_{\eta\to 0}, \end{aligned} \quad (9)$$

where the expectation is with respect to the theory of Eq. (1). Here $K(\eta)$ is obtained by

$$K(i\omega_n) = \frac{g}{4}\int_0^\beta d\tau_1 d\tau_2 \frac{e^{i\omega_n\tau_{12}} - 1}{i\beta\omega_n}\alpha(\tau_{12})e^{i[\phi(\tau_2)-\phi(\tau_1)]}$$

followed by the analytic continuation $i\omega_n \to \eta + i0^+$, which is standard. Equation (9) unequivocally determines the renormalization of the SET. To see this, we notice first that by expanding the effective action of Eq. (8) in powers of ω_n we essentially treat the discrete variable ω_n as a continuous one. This means that the quantities g' and q'

with $|q'| < 1/2$ and $1/E'_c$ in Eqs. (9) are, by construction, a measure for the response of the system to infinitesimal changes in the boundary conditions. This observation immediately leads to a general criterion for the strong coupling *Coulomb blockade phase* of the SET that the perturbative results of Eq. (6) could not give. More specifically, the general statement which says that the SET scales toward an *insulator* as $T \to 0$ implies that the response quantity g' and the fractional part of q' as well as the dimensionless quantity $1/\beta E'_c$ all render equal to zero except for corrections that are exponentially small in β. Since the expressions of Eqs. (9) are all invariant under the shift $q \to q + k$ and $q' \to q' + k$ for integer k, we conclude that the AES theory on the strong coupling side generally displays the *Coulomb blockade* with the novel quantity q', unlike the averaged charge Q on the island, now identified as the *robustly quantized* quasiparticle charge of the SET. This quantization phenomenon, which is depicted in Fig. 1 by the infrared stable fixed points located at integer values $q' = k$, is fundamentally different from the semiclassical picture of the Coulomb blockade since it elucidates the discrete nature of the electronic charge which is independent of tunneling.

Before embarking on the details of scaling, it is important to emphasize that Eqs. (8) and (9) are precisely the same quantities that one normally would obtain in ordinary linear response theory. For example, from the microscopic origins of the AES model, one recognizes g' as the response quantity relating a small potential difference V between the reservoirs to the averaged current $\langle I \rangle$ across the island according to $\langle I \rangle = e^2 GV/h$, where $G = g_l g_r g'/(g_l + g_r)^2$ and h is Planck's constant [14,15]. Similarly, one splits the new quantity q' in Eq. (9) into a thermodynamic piece $Q = q - (2E_c)^{-1}\partial\Omega/\partial q$ which is the average charge on the island and a nonequilibrium piece which is related to the current noise [16]

$$q' = Q - \frac{(g_l + g_r)^2}{2g_l g_r} i \frac{\partial}{\partial V} \int_{-\infty}^{0} dt \langle [I(0), I(t)] \rangle|_{V=0}. \quad (10)$$

The difference between Q and q' is precisely the *antisymmetric* current-current correlation that has attracted a considerable amount of interest over the years [17]. Finally, it can be shown that the quantity E'_c defines the relation $\partial\langle I(\omega)\rangle/\partial\omega = ie^2 V(\omega)/(2E'_c)$, with ω denoting the external frequency.

Weak coupling phase.—By evaluating Eqs. (9) in powers of $1/g$, one obtains the same lowest order results as in Eq. (6). To establish the renormalization of q', one needs the nonperturbative effects of instantons. Following Ref. [8], the universal $\beta = \beta(g', q')$ and $\gamma = \gamma(g', q')$ functions are computed to be [18]

$$\beta_g = \frac{dg'}{d\ln\lambda} = -2 - \frac{4}{g'} - Dg'^2 e^{-g'/2} \cos 2\pi q', \quad (11)$$

$$\beta_q = \frac{dq'}{d\ln\lambda} = -\frac{D}{4\pi} g'^2 e^{-g'/2} \sin 2\pi q', \quad (12)$$

$$\gamma = \frac{d\ln E'_c}{d\ln\lambda} = -\frac{2}{g'} + \frac{D}{2} g'^2 e^{-g'/2} \cos 2\pi q'. \quad (13)$$

Here D is the same as in Eq. (5), and we have included in Eq. (11) the perturbative contribution of order $1/g'$ [19]. The results indicate that instantons are the fundamental objects of the theory that facilitate the *crossover* between the metallic phase with $g' \gg 1$ at high T and the Coulomb blockade phase with $g' \lesssim 1$ that generally appears at a much lower T only.

Strong coupling phase.—We next evaluate Eqs. (9) in terms of a strong coupling expansion about the theory with $g = 0$ [2,20]. Remarkably, this expansion is in many respects the same as the one recently reported for the two-dimensional CP^{N-1} model with large values of N [21]. The results for small values of g' and $u' = q' - k - 1/2$ can be written as follows [18]:

$$\beta_g = -\frac{g'^2}{\pi^2}, \qquad \beta_q = u'\left(1 - \frac{g'}{\pi^2}\right), \qquad \gamma = O(g'^2), \quad (14)$$

indicating that $u' = g' = 0$ is the *critical* fixed point of the AES theory, with g' a marginally irrelevant scaling variable. Equation (14), together with the weak coupling results of Eqs. (11)–(13), is the main justification of the unifying scaling theory illustrated in Fig. 1. To make contact with the existing strong coupling analysis of g' [4,7,15], we employ the basic principles of the renormalization group and obtain the general scaling results $g' = g'(X, Y)$ and $q' = q'(X, Y)$, where [18]

$$X = \tilde{E}_c \tilde{u}/(T\tilde{g}), \qquad Y = (T/\tilde{E}_c) e^{-\pi^2/\tilde{g}}, \quad (15)$$

with $\tilde{u}$, $\tilde{g}$, and $\tilde{E}_c$ denoting the renormalization group starting point (which, by the way, is slightly different from the bare theory u, g, and E_c). An explicit computation gives $g'(0, Y) = |\ln Y|^{-1}$, indicating that the maximum of g' decreases with T like $|\ln T|^{-1}$. Similarly, we find $q'(X, Y) = k + 1/2 - X|\ln Y|^{-1}$, indicating that the width δV_g of the transition with varying $V_g \propto q$ vanishes with T according to $\delta V_g \propto T|\ln T|$ [15].

Critical correlations.—The critical correlations of the AES theory with $u, g \approx 0$ are most elegantly described by the spin $S = 1/2$ effective action [21]

$$\mathcal{S} = \int (\bar{\psi}\partial_\tau \psi + \Delta S_z) + \frac{g}{4} \int_{12} S_-(\tau_1)\alpha(\tau_{12})S_+(\tau_2). \quad (16)$$

Here $\Delta = -2E_c u \approx 0$, $\bar{\psi}$ and ψ denote two-component fermion fields, $S_z = \bar{\psi}\sigma_z\psi/2$, $S_\pm = \bar{\psi}(\sigma_x \pm i\sigma_y)\psi/2$, and $\sigma_{x,y,z}$ are the Pauli matrices. One identifies $Q = k + 1/2 + \langle S_z \rangle$, whereas the operators $S_\pm$ in Eq. (16) have the same meaning as the AES operators $e^{\pm i\phi(\tau)}$ in Eq. (1) that create (annihilate) a unit charge in the SET at time τ. In the absence of tunneling $g = 0$ and at $T = 0$ one has $\langle S_z \rangle = \Delta/(2|\Delta|)$, indicating that the transition at $\Delta = 0$ is a first-order one. Moreover,

$$\langle S_-(0)S_+(\tau)\rangle = \vartheta(\tau\Delta/|\Delta|)\exp(-\Delta\tau), \tag{17}$$

where $\vartheta(x)$ is the Heaviside step function. These results are precisely in accordance with the semiclassical picture of the SET, where $|\Delta|$ denotes the continuously vanishing energy gap between the states $q' = Q = k$ and $k + 1$ of the island as one approaches the critical point. At finite g, the average charge on the island is no longer quantized $Q = k + 1/2 + \tilde{u}/(2\gamma^2|\tilde{u}|)$, with $\gamma = [1 - (\tilde{g}/2\pi^2) \times \ln|\tilde{u}|]^{1/2}$ [7]. On the other hand, Eq. (17) with renormalized operators $S'_\pm$ retains its validity provided Δ is replaced by $\Delta' = \tilde{\Delta}/\gamma^2$. The renormalized energy gap Δ' still vanishes at the critical point, but the states of the SET are now labeled by $q' = k$ and $k + 1$ rather than Q.

In summary, based on the new concept of θ or q' renormalization, we assign a universal significance to the Coulomb blockade in the SET that previously did not exist beyond the semiclassical picture. We have shown that the AES model is, in fact, an extremely interesting and exactly solvable example of a θ vacuum that displays all of the superuniversal topological features that have arisen before in the context of the quantum Hall liquids [8] as well as quantum spin liquids [22]. These include not only the existence of *gapless* or critical excitations at $q' = k + 1/2$ (or $\theta = \pi$) but also the *robust* topological quantum numbers that explain quantization of the electronic charge in the SET at finite values of g. Unlike the conventional theories of the θ angle, however, the strong coupling behavior of the AES model can be studied analytically, and the novel quantity q' should, in general, be taken as an experimental challenge. Experimental designs to probe the antisymmetric current-current correlation in Eq. (10) have already been proposed [23], and recently measurements have been taken from a number of electronic quantum devices [24].

Notice that, when the number of channels in the tunneling contacts are finite rather than infinite, the transition in the SET is likely to become a *second-order* one with a finite value of g' [7]. This closely resembles the more complicated physics of the quantum Hall effect [8]. Finally, the AES theory is known to map onto the "circular brane" model [25] such that the findings of this Letter apply to the latter theory as well. It should be mentioned that physical objectives similar to ours have recently been pursued in Ref. [26] using otherwise heuristic arguments. The reported ideas and conjectures, however, are in many ways in conflict with the present theory.

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