Membranes on an Orbifold

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We provide an M theory interpretation of the recently discovered $\mathcal{N} = 8$ supersymmetric Chern-Simons theory with $SO(4)$ gauge symmetry. The theory is argued to describe two membranes moving in the orbifold $\mathbf{R}^8/\mathbf{Z}_2$. At level $k = 1$ and $k = 2$, the classical moduli space $\mathcal{M}$ coincides with the infrared moduli space of $SO(4)$ and $SO(5)$ super Yang-Mills theory, respectively. For higher Chern-Simons level, the moduli space is a quotient of $\mathcal{M}$. At a generic point in the moduli space, the massive spectrum is proportional to the area of the triangle formed by the two membranes and the orbifold fixed point.

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Introduction.—The eleven dimensional quantum theory of gravity, known as M theory, that unifies the various perturbative string theories was first postulated more than a decade ago [1,2]. The underlying degrees of freedom of M theory include membranes, known as M2-branes. The dynamics of a single M2-brane was first described in [3]; however, the interactions between multiple M2-branes are still far from understood. These interactions are expected to involve novel microscopic degrees of freedom and shed light on the nature of M theory.

At present, the only description of the low-energy dynamics of M2-branes is in terms of the strongly coupled fixed point of a Yang-Mills theory in $d = 2 + 1$ dimensions. The resulting conformal field theory is expected to be invariant under 16 supersymmetries and an $SO(8)$ R symmetry. Beyond these symmetries, little is known of the properties of the fixed point. Progress was made recently in [4], where a novel, conformally invariant, Lagrangian in $d = 2 + 1$ dimensions was constructed which exhibits all the expected symmetries of the problem. The existence of this Lagrangian has come as a surprise to many, for it flies in the face of the well-known "folk-theorem" that the only maximally supersymmetric Lagrangian is super-Yang-Mills. It is to be hoped that the Lagrangian discovered in [4] will prove as important as super-Yang-Mills in elucidating various duality symmetries in string theory and gauge theories, and will pave the way to a better understanding of M theory.

Various aspects of the theory were anticipated in [5–7], and a number of recent papers have explored some of its properties [8–12]. Yet, so far the interpretation in terms of M2-branes has remained somewhat murky. The purpose of this short Letter is to shed some light on this issue through a study of the classical vacuum moduli space and spectrum of the theory. We work with the simplest—and, to date, only—explicit example of the Lagrangian, which is based on an $SO(4)$ gauge symmetry with an integer valued coupling constant k. We will show that, at levels $k = 1$ and $k = 2$, the classical moduli space $\mathcal{M}$ coincides with the infrared limit of $SO(4)$ and $SO(5)$ super Yang-Mills theory. This describes two membranes moving in the background of the orbifold $\mathbf{R}^8/\mathbf{Z}_2$, without and with discrete torsion, respectively. For $k > 2$, we find that the vacuum moduli space is the quotient of $\mathcal{M}$. The group acts on the moduli space, but does not appear to have a natural action on the underlying spacetime. We further show that, at a generic point in the moduli space, the mass of the heavy states is proportional to the area of the triangle formed by the two membranes and the fixed point, and we make some comments on the implications of this mass formula.

The M2-brane Lagrangian.—The Lagrangian presented in [4] is built around a 3-algebra $\mathcal{A}$. This is a vector space with basis T^a, $a = 1, \ldots, \dim\mathcal{A}$, endowed with a trilinear antisymmetric product $[T^a, T^b, T^c] = f^{abc}_d T^d$. The algebra is accompanied by an inner product, $h^{ab} = \mathrm{Tr}(T^a T^b)$, with which indices may be raised and lowered. The structure constants of the algebra are then required to be totally antisymmetric, $f^{abcd} = f^{[abcd]}$, and to satisfy the "fundamental identity"

$$f^{aef}_g f^{bcdg} - f^{bef}_g f^{acdg} + f^{cef}_g f^{abdg} - f^{def}_g f^{abcg} = 0.$$

The matter fields consist of 8 algebra-valued scalar fields X^I_a, $I = 1, \ldots, 8$, transforming in the $\mathbf{8}_v$ of $SO(8)$, together with algebra-valued spinors Ψ^a transforming in the $\mathbf{8}_s$ of $SO(8)$. The theory also includes a nonpropagating gauge field A^{ab}_μ. The bosonic dynamics is governed by the Lagrangian,

$$\mathcal{L} = -\frac{1}{2}\mathcal{D}^\mu X^{Ia}\mathcal{D}_\mu X^I_a - V(X) + \frac{1}{2}\epsilon^{\mu\nu\lambda}\Big(f_{abcd}A^{ab}_\mu \partial_\nu A^{cd}_\lambda + \frac{2}{3}f_{cda}{}^g f_{efgb}A^{ab}_\mu A^{cd}_\nu A^{ef}_\lambda\Big), \quad (1)$$

where the scalar potential is

$$V(X) = \frac{1}{12}f_{abcd}f_{efgd}X^{Ia}X^{Jb}X^{Kc}X^{Ie}X^{Jf}X^{Kg}, \quad (2)$$

while the covariant derivative is defined by $\mathcal{D}_\mu X^{Ia} = \partial_\mu X^{Ia} + f^a_{bcd}A^{cd}_\mu X^{Ib}$. The theory is invariant under 16 supercharges and the gauge symmetry: $\delta X^I_a = -f_{abcd}\Lambda^{bc}X^{Id}$ and $\delta A^{ab}_\mu = f_{abcd}\mathcal{D}_\mu \Lambda^{ab}$. Presently, the only known, finite-dimensional, representation of a 3-algebra has

$\dim\mathcal{A} = 4$, and the gauge field A_μ^{ab} is valued in $so(4)$. The inner product is taken to be $h^{ab} = \delta^{ab}$ while the structure constants are [4]

$$f^{abcd} = \frac{2\pi}{k}\epsilon^{abcd}. \tag{3}$$

The requirement that the theory is invariant under large gauge transformations imposes the usual quantization on the Chern-Simons coefficient: $k \in \mathbf{Z}$. In the rest of this Letter, we study a few elementary aspects of this $SO(4)$ theory.

The classical moduli space.—We start by examining the vacuum moduli space of the classical theory, defined as solutions to $V(X) = 0$ modulo gauge transformations. This was previously discussed in [8,11]. However, in both analyses, there was no obvious interpretation of the moduli space in terms of known M theoretic objects.

By a suitable gauge transformation, solutions to $V(X) = 0$ may be written as [8]

$$X^I = r_1^I T^1 + r_2^I T^2. \tag{4}$$

However, as stressed in [11], there are additional gauge symmetries which preserve the form of X^I. Since X transforms in the fundamental representation of $SO(4)$, we may act by $g \in SO(4)$ in the block diagonal form

$$g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}, \tag{5}$$

where $g_1, g_2 \in O(2)$ with $\det g_1 = \det g_2$. Let us first look at a number of discrete symmetries. Since g_2 acts trivially on (4), we can effectively ignore it and simply look at $g_1 \in O(2, \mathbf{Z})$. There are three choices for g_1 which generate all of $O(2, \mathbf{Z})$ and act on r_1 and r_2 as

$$\begin{aligned} &\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}: r_1 \to -r_1, \qquad r_2 \to r_2 \\ &\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}: r_1 \to r_1, \qquad r_2 \to -r_2 \\ &\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}: r_1 \to r_2, \qquad r_2 \to r_1. \end{aligned} \tag{6}$$

We have still to divide out by the continuous $g_1 \in SO(2) \cong U(1)_{12}$ symmetry, which acts as $z^I \to e^{i\theta} z^I$ where $z^I = r_1^I + i r_2^I$. If we make use of all three discrete gauge symmetries (6), we already have the identification $z^I \to i z^I$. Thus, in order not to overreact, we must take the parameter θ to have range $\theta \in [0, \pi/2)$. Alternatively, we could impose just one discrete symmetry, say the last one which reads $z \to i\bar{z}$, and take $\theta \in [0, \pi]$.

Dividing out by this continuous gauge symmetry would seem to leave us with a 15-dimensional moduli space. This is a rather odd state of affairs and would contradict the expectations of supersymmetry. We will now show that by considering the unbroken gauge symmetry of the theory, we will recover this lost dimension of moduli space. To see this, we proceed by writing down the low-energy effective action.

Because of the ϵ^{abcd} appearing in the covariant derivative, the $U(1)_{12}$ gauge symmetry is associated to the gauge field A_μ^{34}. Normalizing so that z^I has charge $+1$, we define $B_\mu = \frac{4\pi}{k} A_\mu^{34}$. Then, the kinetic terms on moduli space are given by

$$\mathcal{L}_{\text{moduli}} = -\frac{1}{2}|\mathcal{D}_\mu z^I|^2, \tag{7}$$

with $\mathcal{D}z = \partial z + iBz$. At a generic point in moduli space, there is also an unbroken $SO(2)$ symmetry [11], arising from the action g_2 in (5). We will call this symmetry $U(1)_{34}$. It is associated to the gauge field $C_\mu = \frac{4\pi}{k} A_\mu^{12}$, where the normalization is again taken to ensure that charged fields have charge ± 1 under C_μ. A mixed Chern-Simons term couples the B and C gauge fields:

$$\mathcal{L}_{cs} = \frac{k}{2\pi}\epsilon^{\mu\nu\lambda} B_\mu \partial_\nu C_\lambda. \tag{8}$$

It was shown in [9] that integrating out the broken gauge field B induces a Maxwell term for C, promoting it to a dynamical field. (In fact, the calculation in [9] was done at a nongeneric point in moduli space with an unbroken $SU(2)$ gauge symmetry, but it proceeds in the same manner at a generic point.) Here, we instead replace the unbroken gauge field C with its dual photon, introduced in its usual guise as a Lagrange multiplier to impose the Bianchi identity on the field strength $G_{\mu\nu} = \partial_\mu C_\nu - \partial_\mu C_\nu$.

$$\mathcal{L}_{\text{dual}} = -\frac{1}{8\pi}\sigma\epsilon^{\mu\nu\lambda}\partial_\mu G_{\nu\lambda}. \tag{9}$$

The normalization is chosen such that $\sigma \in [0, 2\pi)$. To see this, note that $U(1)_{34} \subset SU(2)_{\text{diag}} \subset SO(4)$, with all matter fields in our theory living in the adjoint of $SU(2)_{\text{diag}}$. The magnetic configurations of the theory are therefore given by the familiar Euclidean 't Hooft-Polyakov monopole solutions which satisfy the quantization condition,

$$\frac{1}{8\pi}\int d^3x \epsilon^{\mu\nu\lambda}\partial_\mu G_{\nu\lambda} \in \mathbf{Z}. \tag{10}$$

In the presence of the mixed Chern-Simons term (8), the shift symmetry of the dual photon becomes gauged under $U(1)_{12}$. This follows because the topological current ${}^*G$, which generates the shift symmetry of the dual photon, is coupled to B_μ. It is also simple to see by collecting together the various pieces of the Lagrangian, which can be found in (7)–(9): $\mathcal{L} = \mathcal{L}_{\text{moduli}} + \mathcal{L}_{CS} + \mathcal{L}_{\text{dual}}$. This is invariant under the gauge action $U(1)_{12}$

$$z^I \to e^{i\theta} z^I, \qquad \sigma \to \sigma + 2k\theta, \qquad B_\mu \to B_\mu - \partial_\mu\theta$$

together with the discrete gauge symmetries (6), which now also induce a sign flip for σ.

We can go further and eliminate the field strength $G_{\mu\nu}$. Since it is now unconstrained by the Bianchi identity, it acts as a Lagrange multiplier imposing the requirement that $B_\mu = -(1/2k)\partial_\mu \sigma$ is pure gauge. This results in the action

$$\mathcal{L} = -\frac{1}{2}\left|\partial_\mu z^I - \frac{i}{2k} z^I \partial_\mu \sigma\right|^2, \tag{11}$$

and we observe that σ can be eliminated by the field redefinition $z^I \to e^{-i\sigma/2k} z^I$. However, this transformation still leaves us with a number of discrete identifications which we now examine more carefully.

The theory at level $k = 1$ and $k = 2$.—For $k = 1$, we impose just one of the discrete symmetries, which we take to be $z \to i\bar{z}$, with $\theta \in [0, \pi]$. We can now fix the $U(1)_{12}$ gauge symmetry by imposing $\sigma = 0$, leaving us with remnant $\mathbf{Z}_2$ which acts by $\sigma \to \sigma + 2\pi$ and $z \to -z$. The moduli space at level $k = 1$ is thus,

$$\mathcal{M}_{k=1} \cong \frac{\mathbf{R}^8 \times \mathbf{R}^8}{\mathbf{Z}_2 \times \mathbf{Z}_2}. \tag{12}$$

Writing $z = r_1 + ir_2$, the two $\mathbf{Z}_2$ factors act as $(r_1, r_2) \to (-r_1, -r_2)$ and $(r_1, r_2) \to (r_2, r_1)$. As observed in [13], this coincides with the infrared limit of the moduli space of $d = 2 + 1$ dimensional, maximally supersymmetric Yang-Mills (SYM) theory with an $SO(4)$ gauge group.

For $k = 2$, we may again fix the $U(1)_{12}$ gauge symmetry by setting $\sigma = 0$. Imposing all three discrete symmetries, we have $\theta \in [0, \pi/2)$ which now leaves no further residual transformation. The moduli space dynamics is simply given by the 8 complex scalars z^I, endowed with a flat metric and subject to the discrete symmetries (6). We conclude that the classical vacuum moduli space of the theory at level $k = 2$ is

$$\mathcal{M}_{k=2} \cong \frac{(\mathbf{R}^8/\mathbf{Z}_2) \times (\mathbf{R}^8/\mathbf{Z}_2)}{\mathbf{Z}_2}. \tag{13}$$

This coincides with the moduli space of $SO(5)$ SYM theory in the infrared limit or, alternatively, the configuration space of two M2-branes in the background of the orbifold $\mathbf{R}^8/\mathbf{Z}_2$.

We strike a note of caution: the $k = 1$ and $k = 2$ theories are strongly coupled at all points in their moduli space. Nonetheless, we will assume that we can take (12) and (13) at face value. We take this as evidence that the $k = 1$ and $k = 2$ theories describe the infrared fixed point of $SO(4)$ and $SO(5)$ SYM, respectively.

Let us briefly review a few pertinent facts about the M theory orbifold $\mathbf{R}^8/\mathbf{Z}_2$. There are actually two different such orbifolds, distinguished by discrete torsion for G_4 arising because $H^4(\mathbf{RP}^7, \mathbf{Z}) \cong \mathbf{Z}_2$ [14]. The orbifolds with and without torsion are referred to as type-B and type-A, respectively. The low-energy dynamics of N M2-branes in these orbifold backgrounds is thought to be governed by a maximally supersymmetric, $SO(8)$ invariant conformal fixed point. These arise as the strong coupling limit of maximally supersymmetric Yang-Mills (SYM) in $d = 2 + 1$ dimensions with gauge groups $O(2N)$, $SO(2N + 1)$, and $Sp(N)$. As explained in [14,15], the fact that these three classical groups flow to one of only two possible theories implies nontrivial IR dualities between distinct UV theories. The RG flows occur as follows: $O(2N)$ SYM flows to the theory on M2-branes on the A-type orbifold; $SO(2N + 1)$ SYM flows to the theory on the B-type orbifold; while $Sp(N)$ SYM flows to either the theory on the A-type or B-type orbifold, depending on the expectation value of the dual photon. Comparing to our previous analysis, we see that the $k = 1$ theory describes two membranes on the A-type orbifold, while the $k = 2$ theory describes two membranes on the B-type orbifold.

The identification of the M2-brane Lagrangian (1) with M2-branes on an orbifold also resolves a puzzle raised in [11] regarding chiral primary operators. The bosonic, gauge invariant operators of (1) live in tensor representations of $SO(8)$ with an even number of indices. Yet the chiral primary operators derived from M theory on $\mathrm{AdS}_4 \times \mathbf{S}^7$ live in the symmetric traceless s-index representations of $SO(8)$, with both even and odd s. However, pleasingly only the even s representations survive the orbifold projection in supergravity [16]. Although the AdS/CFT analysis is valid only at large N, it is comforting that this basic feature agrees with the $N = 2$ M2-brane theory.

The theory at level $k > 2$.—Perhaps the most intriguing consequence of the Lagrangian (1) is the existence of a weakly coupled limit when $k \gg 1$. Understanding how such a limit arises from an M theoretic description may be our best hope of getting a handle on the underlying microscopic degrees of freedom.

For $k > 2$, setting $\sigma = 0$ does not completely fix the $U(1)_{12}$ gauge action (10). There exists a residual $\mathbf{Z}_k$ symmetry which leaves $\sigma = 0 \bmod 2\pi$ and is generated by $z^I \to e^{i\pi/k} z^I$. As pointed out in [13], this $\mathbf{Z}_k$ action does not commute with the $\mathbf{Z}_2$ actions of Eq. (6). Between them, they generate the dihedral group D_{2k}. We conclude that the moduli space is given by

$$\mathcal{M}_k \cong \frac{\mathbf{R}^8 \times \mathbf{R}^8}{D_{2k}}. \tag{14}$$

However, while the group D_{2k} has a simple action on the moduli space, it does not appear to have such a description on the spacetime transverse to the M2-branes for $k > 2$. In particular, it does not leave the distances between branes fixed. Needless to say, it would be potentially rather interesting to better understand the microscopic meaning of this quotient action and these higher k theories. A curious observation of [13] is that the moduli space for $k = 3$ coincides with the infrared limit of SYM with G_2 gauge group.

The spectrum and non-Abelian gauge restoration.—Since the $\mathbf{Z}_k$ action $z^I \to e^{i\pi/k} z^I$ does not preserve the distance between the two branes, it would not make much sense for a pair of D-branes. In string theory, this distance dictates the spectrum of massive states arising from stretched strings. Yet the M2-brane theory appears to be blind to the transverse distance between the two branes. It knows only about transverse areas. This is clear if we look at the classical mass spectrum, which we trust for $k \gg 1$. Sitting at a generic point in moduli space, we may employ the $SO(8)$ R symmetry to rotate the M2-branes to lie in the $X^7 - X^8$ plane. Then, the mass of states is given by

$$M = \frac{4\pi}{k} A \tag{15}$$

where $A = \frac{1}{2}|r_1^7 r_2^8 - r_1^8 r_2^7| = \frac{1}{4}|\bar{z}^7 z^8 - \bar{z}^8 z^7|$ is the area of the triangle formed by the two M2-branes and the orbifold fixed point. This is manifestly invariant under the $\mathbf{Z}_k$ action.

This mass formula implies that new states become massless when the branes become colinear with the orbifold fixed point. This is to be contrasted with the familiar statement that states on D-branes become massless when branes coincide. In generic vacua, the R symmetry is broken to $SO(6)$ and, as we noted previously, a $U(1)_{34}$ gauge symmetry survives. However, when the branes are colinear, and the R symmetry is broken to $SO(7)$, a full $SO(3)$ gauge symmetry is left unbroken. This was the situation examined in [9] where it was shown that, upon integrating out the broken gauge generators, this $SO(3)$ gauge field becomes dynamical. These are the new massless states.

The emergence of this dynamical $SO(3)$ gauge field is something of a blessing, for it removes a potential difficulty in interpreting the expectation value (4) as the position of two branes. The problem is that whenever the branes are colinear, one can change the relative positions of the branes through a gauge transformation. For example, the $SO(7)$ preserving expectation values

$$X^I = r^I(c_1 T^1 + c_2 T^2) \tag{16}$$

are gauge equivalent for all c_1 and c_2 such that $c_1^2 + c_2^2$ is constant. Naively, this would equate configurations with different separations between colinear branes and the fixed point. In fact, the theory does distinguish between these configurations, but it is somewhat hard to see explicitly. The presence of the dynamical, unbroken $SO(3)$ gauge field means that there is a non-Abelian dual photon, whose expectation value will determine the relative positions of the branes. This is entirely analogous to the situation of two D2-branes in IIA string theory, for which the moduli space is $(\mathbf{R}^7 \times \mathbf{S}^1)/\mathbf{Z}_2$. Even at the origin of $\mathbf{R}^7$, where the gauge group is unbroken, the branes may still be separated in a nonsingular fashion along the M theory circle. However, seeing how this explicitly arises from the non-Abelian dual photon is difficult.

A related fact is that the appearance of the massless states when the branes are colinear does not necessarily imply a singularity in the low-energy effective theory. This is exemplified in the D2-brane, where there are only isolated singularities in the moduli space, rather than a whole $\mathbf{S}^1$'s worth of singularities at the origin of $\mathbf{R}^7$. Indeed, from the M theory perspective, the generic point with colinear branes should be smooth. More precisely, we expect that, in the vacua (16), there is just a single singularity for the $k = 1$ theory, corresponding to the two branes sitting on top of each other. For the $k = 2$ theory, there should be two singularities, the first corresponding to the two branes sitting on top of each other, while the second corresponds to one brane sitting on the orbifold fixed plane which is now expected to result in a nontrivial fixed point.

Finally, it is tempting to believe that the mass formula (15) is hinting at some fundamental degree of freedom of M theory. The fact that the mass should scale as an area is, for $k \gg 1$, a consequence of conformal invariance, and the triangle is the only natural area in the theory. Nonetheless, the appearance of such a ''3-pronged'' object is intriguing, not least because such states would naively explain the famous N^3 entropy of the M5-brane theory [17]. However, quite how one could scale such states to account for the $N^{3/2}$ entropy for M2-branes, in a controllable weakly coupled regime, appears as tantalizingly mysterious as ever.

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