Building a Holographic Superconductor

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We show that a simple gravitational theory can provide a holographically dual description of a superconductor. There is a critical temperature, below which a charged condensate forms via a second order phase transition and the (dc) conductivity becomes infinite. The frequency dependent conductivity develops a gap determined by the condensate. We find evidence that the condensate consists of pairs of quasiparticles.

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Introduction.—A remarkable result to emerge from string theory is the AdS/CFT correspondence [1], which relates string theory on asymptotically anti-de Sitter spacetimes to a conformal field theory on the boundary. The correspondence provides an established method for calculating correlation functions in a strongly interacting field theory using a dual classical gravity description [2]. Transport quantities are extracted by solving Einstein's equations in the dual theory. This has lead to successful applications of AdS/CFT to nuclear physics, in particular, the results of heavy ion collisions at RHIC [3]. Connections to condensed matter systems are being explored; phenomena such as the Hall effect [4] and Nernst effect [5] have dual gravitational descriptions. Here we exhibit a dual gravitational description of superconductivity.

Conventional superconductors, including many metallic elements (Al, Nb, Pb, ...), are well described by BCS theory [6]. However, basic aspects of unconventional superconductors, including the pairing mechanism, remain incompletely understood. There are many indications that the normal state in these materials is not described by the standard Fermi liquid theory [7]. Although there is a long way to go before making sharp connections between our results and actual experimental systems, we hope that a tractable theoretical model of a strongly coupled system which develops superconductivity will be of interest. Several important unconventional superconductors, such as the cuprates and organics, are layered and much of the physics is $2 + 1$ dimensional. Our model will also be $2 + 1$ dimensional.

To map a superconductor to a gravity dual, we introduce temperature by adding a black hole [8] and a condensate through a charged scalar field. To reproduce the superconductor phase diagram, we require a system that admits black holes with scalar hair at low temperature, but no hair at high temperature. While neutral AdS black holes can have neutral scalar hair only if the theory is unstable [9], Gubser has recently suggested that a charged black hole will support charged scalar hair if the charges are large enough [10]. We consider a simpler version of Gubser's bulk theory and show that it indeed provides a dual description of a superconductor.

The model: condensing charged operators.—We start with the planar Schwarzschild–anti-de Sitter black hole

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(dx^2 + dy^2), \qquad (1)$$

where $f = \frac{r^2}{L^2} - \frac{M}{r}$. Here L is the AdS radius and M determines the Hawking temperature of the black hole: $T = \frac{3M^{1/3}}{4\pi L^{4/3}}$. This black hole is $3 + 1$ dimensional, and so will be dual to a $2 + 1$ dimensional theory. In this background, we now consider a Maxwell field and a charged complex scalar field, with Lagrangian density [11]

$$\mathcal{L} = -\tfrac{1}{4}F^{ab}F_{ab} - V(|\Psi|) - |\nabla\Psi - iA\Psi|^2. \qquad (2)$$

For concreteness, we will focus on the case $V(|\Psi|) = -\frac{2|\Psi|^2}{L^2}$. The negative mass squared is above the Breitenlohner-Freedman bound [12] and hence is stable. It corresponds to a conformally coupled scalar in our background (1) and arises in several contexts in which the AdS_4/CFT_3 correspondence is embedded into string theory. For instance, the truncation of M theory on $AdS_4 \times S^7$ to $\mathcal{N} = 8$ gauged supergravity has scalars and pseudoscalars with this mass, dual to the bilinear operators $\mathrm{tr}\Phi^{(I}\Phi^{J)}$ and $\mathrm{tr}\Psi^{(I}\Psi^{J)}$ in the dual $\mathcal{N} = 8$ Super Yang-Mills theory, respectively. However, we should note that our Lagrangian (2) has yet to be obtained from M theory. It is a minimal phenomenological model. We expect that our choice of mass is not crucial, and qualitatively similar results will hold, e.g., for massless fields.

We will work in a limit in which the Maxwell field and scalar field do not backreact on the metric. This limit describes fields that are small in Planck units. This decoupled Abelian-Higgs sector can be obtained from the full Einstein-Maxwell-scalar theory of [10] through a scaling limit in which the product of the charge of the black hole and the charge of the scalar field is held fixed while the latter is taken to infinity. Thus we will obtain solutions of nonbackreacting scalar hair on the black hole. Our simple model captures the physics of interest.

Taking a plane symmetric ansatz, $\Psi = \Psi(r)$, the scalar field equation of motion is

$$\Psi'' + \left(\frac{f'}{f} + \frac{2}{r}\right)\Psi' + \frac{\Phi^2}{f^2}\Psi + \frac{2}{L^2 f}\Psi = 0, \qquad (3)$$

where the scalar potential $A_t = \Phi$. With $A_r = A_x = A_y = 0$, the Maxwell equations imply that the phase of Ψ must be constant. We take Ψ to be real. The equation for the scalar potential Φ is the time component of the equation of motion for a massive vector field

$$\Phi'' + \frac{2}{r}\Phi' - \frac{2\Psi^2}{f}\Phi = 0, \qquad (4)$$

where $2\Psi^2$ is the, in our case, r dependent mass. The charged condensate has triggered a Higgs mechanism in the bulk theory. At the horizon, $r = r_0$, for Φdt to have finite norm, $\Phi = 0$, and (3) then implies $\Psi = -3r_0\Psi'/2$. Thus, there is a two parameter family of solutions which are regular at the horizon. Integrating out to infinity, these solutions behave as

$$\Psi = \frac{\Psi^{(1)}}{r} + \frac{\Psi^{(2)}}{r^2} + \cdots, \qquad \Phi = \mu - \frac{\rho}{r} + \cdots. \qquad (5)$$

For Ψ, both of these falloffs are normalizable [13], so one can impose the boundary condition that either one vanishes [14]. Imposing the condition that either $\Psi^{(1)}$ or $\Psi^{(2)}$ vanish leaves a one parameter family of solutions.

Properties of the dual field theory can be read off from the asymptotic behavior of the solution. The asymptotic behavior (5) of Φ yields the chemical potential μ and charge density ρ of the field theory. The condensate of the scalar operator $\mathcal{O}$ in the field theory dual to the field Ψ is given by

$$\langle \mathcal{O}_i \rangle = \sqrt{2}\Psi^{(i)}, \qquad i = 1, 2 \qquad (6)$$

with the boundary condition $\epsilon_{ij}\Psi^{(j)} = 0$. The $\sqrt{2}$ normalization simplifies subsequent formulae. Note that $\mathcal{O}_i$ is an operator with dimension i. From this point on we will work in units in which the AdS radius is $L = 1$. Recall that T has mass dimension one, and ρ has mass dimension two so $\langle \mathcal{O}_i \rangle / T^i$ and ρ/T^2 are dimensionless quantities.

An exact solution to Eqs. (3) and (4) is clearly $\Psi = 0$ and $\Phi = \mu - \rho/r$. It is difficult to find other analytic solutions to these nonlinear equations. However, it is straightforward to solve them numerically. We find that solutions exist with all values of the condensate $\langle \mathcal{O} \rangle$. As shown in Fig. 1, in order for the operator to condense, a minimal ratio of charge density over temperature squared is required.

The right curve in Fig. 1 is qualitatively similar to that obtained in BCS theory, and observed in many materials, where the condensate goes to a constant at zero temperature. The left curve starts similarly, but at low temperature the condensate appears to diverge as $T^{-1/3}$. When the condensate becomes very large, the backreaction on the bulk metric can no longer be neglected. At extremely low temperatures, we will eventually be outside the region of validity of our approximation.

By fitting these curves, we see that for small condensate there is a square root behavior that is typical of second order phase transitions. Specifically, we find

$$\langle \mathcal{O}_1 \rangle \approx 9.3 T_c (1 - T/T_c)^{1/2}, \quad \text{as} \quad T \to T_c, \qquad (7)$$

where the critical temperature is $T_c \approx 0.226\rho^{1/2}$ and

$$\langle \mathcal{O}_2 \rangle \approx 144 T_c^2 (1 - T/T_c)^{1/2}, \quad \text{as} \quad T \to T_c, \qquad (8)$$

where now $T_c \approx 0.118\rho^{1/2}$. The continuity of the transition can be checked by computing the free energy. Finite temperature continuous symmetry breaking phase transitions are only possible in $2 + 1$ dimensions in the large N limit (i.e., the classical gravity limit of our model), where fluctuations are suppressed. These transitions will become crossovers at finite N.

For $T < T_c$ a charged scalar operator, $\langle \mathcal{O}_1 \rangle$ or $\langle \mathcal{O}_2 \rangle$, has condensed. It is natural to expect that this condensate will lead to superconductivity.

Maxwell perturbations and the conductivity.—We now compute the conductivity in the dual CFT as a function of frequency. This requires us to solve for fluctuations of the vector potential A_x in the bulk. The Maxwell equation at zero spatial momentum and with a time dependence of the form $e^{-i\omega t}$ is

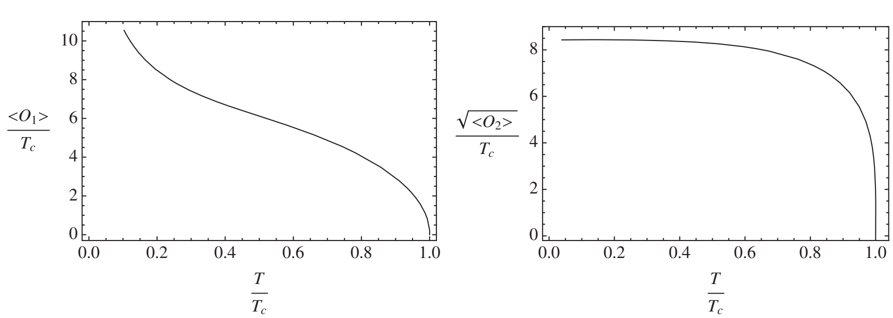

FIG. 1. The condensate as a function of temperature for the two operators $\mathcal{O}_1$ and $\mathcal{O}_2$. The condensate goes to zero at $T = T_c \propto \rho^{1/2}$.

$$A_x'' + \frac{f'}{f}A_x' + \left(\frac{2}{f^2} - \frac{2\Psi^2}{f}\right)A_x = 0. \tag{9}$$

To compute causal behavior, we solve this equation with ingoing wave boundary conditions at the horizon [16]: $A_x \propto f^{-i\omega/3r_0}$. The asymptotic behavior of the Maxwell field at large radius is seen to be

$$A_x = A_x^{(0)} + \frac{A_x^{(1)}}{r} + \cdots. \tag{10}$$

The AdS/CFT dictionary tells us that the dual source and expectation value for the current are given by

$$A_x = A_x^{(0)}, \qquad \langle J_x \rangle = A_x^{(1)}. \tag{11}$$

Now from Ohm's law we can obtain the conductivity

$$\sigma(\omega) = \frac{\langle J_x \rangle}{E_x} = -\frac{\langle J_x \rangle}{\dot{A}_x} = -\frac{i\langle J_x \rangle}{A_x} = -\frac{iA_x^{(1)}}{A_x^{(0)}}. \tag{12}$$

In Fig. 2 we plot the frequency dependent conductivity obtained by solving (9) numerically. The horizontal line corresponds to temperatures at or above the critical value, where there is no condensate. The conductivity in the normal phase is frequency independent in theories with AdS_4 duals [17]. The subsequent curves describe successively lower values of the temperature (for fixed charge density). As the temperature is lowered, a gap opens. The gap becomes increasingly deep until the (real part of the) conductivity is exponentially small.

There is also a delta function at $\omega = 0$ which appears as soon as $T < T_c$. This can be seen by looking at the imaginary part of the conductivity. The Kramers-Kronig relations relate the real and imaginary parts of any causal quantity, such as the conductivity, when expressed in frequency space. One of the relations is

$$\mathrm{Im}[\sigma(\omega)] = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\mathrm{Re}[\sigma(\omega')]d\omega'}{\omega' - \omega}. \tag{13}$$

From this formula we can see that the real part of the conductivity contains a delta function, $\mathrm{Re}[\sigma(\omega)] = \pi\delta(\omega)$, if and only if the imaginary part has a pole, $\mathrm{Im}[\sigma(\omega)] = 1/\omega$. We find that there is indeed a pole in $\mathrm{Im}[\sigma]$ at $\omega = 0$ for all $T < T_c$. The superfluid density is the coefficient of the delta function [18]

$$\mathrm{Re}[\sigma(\omega)] \sim \pi n_s \delta(\omega). \tag{14}$$

By (13), n_s is also the coefficient of the pole in the imaginary part $\mathrm{Im}[\sigma(\omega)] \sim n_s/\omega$ as $\omega \to 0$. We find that the superfluid density vanishes linearly with $T_c - T$:

$$n_s \approx C_i(T_c - T) \quad \text{as} \quad T \to T_c, \tag{15}$$

where $C_1 = 16.5$ and $C_2 = 24$, for the $\mathcal{O}_1$ and $\mathcal{O}_2$ cases.

In Fig. 3 we rescaled the small T/T_c plots of Fig. 2 by plotting the frequency in units of the condensate. The curves tend to a limit in which the width of the gap is proportional to the size of the condensate. The differing shapes of the plots in Fig. 3 are precisely what is expected from type II and type I coherence factors, respectively [6]. Type II coherence suppresses absorption near the edge of the gap, explaining the slower rise of $\mathrm{Re}[\sigma]$ in the left plot. It is possible that this difference is due to the operator $\mathcal{O}_1$ being a pair of bosons and $\mathcal{O}_2$ a pair of fermions, as in the case of $AdS_4 \times S^7$.

The Ferrell-Glover sum rule states that $\int \mathrm{Re}[\sigma]d\omega$ is independent of temperature. The area missing under the curve $\mathrm{Re}[\sigma]$ due to the gap is made up by the delta function at $\omega = 0$. That $\mathrm{Re}[\sigma]$ exceeds the value one in Fig. 3 (right) implies then that the superfluid density n_s must be correspondingly reduced for the $\mathcal{O}_2$ system compared with the $\mathcal{O}_1$ system for $T \ll T_c$.

We can also compute the contribution of the normal, nonsuperconducting, component to the dc conductivity. Let us define

$$n_n = \lim_{\omega \to 0}\mathrm{Re}[\sigma(\omega)]. \tag{16}$$

From our numerics we obtain

$$n_n \sim e^{-\Delta/T}, \quad \text{for} \quad \frac{\Delta}{T} \gg 1, \tag{17}$$

where we have $\Delta = \langle \mathcal{O}_1 \rangle/2$ and $\Delta = \sqrt{\langle \mathcal{O}_2 \rangle}/2$. Numerically this factor of 1/2 is accurate to at least 4%. From (17), Δ is immediately interpreted as the energy gap for charged excitations. The gap we found previously in the frequency dependent conductivity was 2Δ. The extra factor

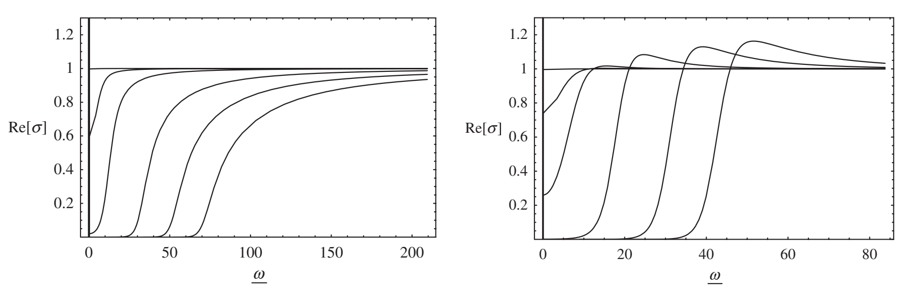

FIG. 2. The formation of a gap in the real, dissipative, part of the conductivity as the temperature is lowered below the critical temperature. Results shown for both the $\mathcal{O}_1$ operator (left) and the $\mathcal{O}_2$ operator (right). There is also a delta function at $\omega = 0$. The rightmost curve in each plot corresponds to $T/T_c = 0.0066$ (left) and $T/T_c = 0.0026$ (right).

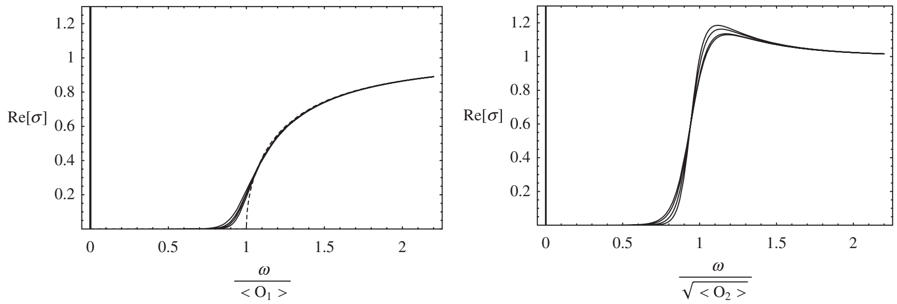

FIG. 3. The gap at small T/T_c, with frequency normalized by the condensate. The dashed curve on the left plot is (18).

of 2 is expected if the gapped charged quasiparticles are produced in pairs, suggesting a "pairing mechanism" is at work in our model. Our results for Δ are suggestive of strong pairing interactions. In Fig. 1 (right) at $T = 0$ we found $2\Delta \approx 8.4T_c$, which might be compared with the BCS prediction $2\Delta \approx 3.54T_c$. The larger value is expected for deeply bound Cooper pairs. Indeed, in Fig. 1 (left), Δ actually diverges at low T.

It is natural to ask if one can reproduce this limiting low temperature behavior by just taking $M = 0$ in our background metric (1). One problem is that there are no solutions to the field equations (3) and (4) which are smooth on the horizon of the Poincaré patch. Nevertheless, for the $\mathcal{O}_1$ case, we have observed numerically that at low temperatures, $\Psi \approx \langle \mathcal{O}_1 \rangle / \sqrt{2} r$. Taking $M \to 0$ where $f \approx r^2$, (9) can be solved exactly to yield $A_x = A_x^{(0)} \exp(\pm \sqrt{\langle \mathcal{O}_1 \rangle^2 - \omega^2}/r)$. This exact result then produces the nonzero conductivities

$$\sigma = \frac{i\sqrt{\langle \mathcal{O}_1 \rangle^2 - \omega^2}}{\omega} \mathrm{sgn}(\langle \mathcal{O}_1 \rangle - \omega) \tag{18}$$

via (12). The curves on the left of Fig. 3 are well approximated by the conductivity (18). We have included (18) as a dashed curve in this plot.

Discussion.—We have shown that a simple 3 + 1 dimensional bulk theory can reproduce properties of a 2 + 1 dimensional superconductor. Below a second order phase transition the dc conductivity becomes infinite and an energy gap for charged excitations is formed.

There are many extensions that we hope to consider elsewhere: (i) By considering spatially varying fields and an external magnetic field, one can compute the superconducting coherence length and penetration depth, respectively. (ii) One would like to consider a wider class of models by allowing for more general masses for the charged scalar field. (iii) The primary consequence of backreaction on the bulk spacetime metric should be a divergence in the dc conductivity, as a constant external field accelerates the charged medium. (iv) Perhaps the most interesting question is to understand the "pairing mechanism" in field theory that leads to a condensate in these systems.

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