Self-Isospectrality, Special Supersymmetry, and their Effect on the Band Structure

Francisco Correa,[1] Vít Jakubský,[1] Luis-Miguel Nieto,[2] and Mikhail S. Plyushchay[1]
[1]*Departamento de Física, Universidad de Santiago de Chile, Casilla 307, Santiago 2, Chile*
[2]*Departamento de Física Teórica, Atómica y Óptica, Universidad de Valladolid, 47071, Valladolid, Spain*
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We study a planar model of a nonrelativistic electron in periodic magnetic and electric fields that produce a 1D crystal for two spin components separated by a half-period spacing. We fit the fields to create a self-isospectral pair of finite-gap associated Lamé equations shifted for a half-period, and show that the system obtained is characterized by a new type of supersymmetry. It is a special nonlinear supersymmetry generated by three commuting integrals of motion, related to the parity-odd operator of the associated Lax pair, that coherently reflects the band structure and all its peculiarities. In the infinite-period limit it provides an unusual picture of supersymmetry breaking.

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Introduction.—Supersymmetry as a fundamental symmetry of Nature still waits for experimental confirmation, but as a kind of symmetry between bosonic and fermionic states, it already turned out to be fruitful in diverse areas, including nuclear [1,2], atomic, solid-state, and statistical physics [3]. Supersymmetric quantum mechanics [3] was introduced under investigation of the problem of supersymmetry breaking in field theory [4]. In the simplest case a system is characterized there by a 2×2 diagonal matrix Hamiltonian, $H = \mathrm{diag}(H^+, H^-)$, and by two antidiagonal matrix integrals of motion (supercharges) Q_1 and $Q_2 = i\sigma_3 Q_1$. Supercharges are first order differential operators generating an $N = 2$ superalgebra

$$\{Q_a, Q_b\} = 2\delta_{ab}H, \qquad [Q_a, H] = 0. \tag{1}$$

Such a system has an additional integral of motion $\Gamma = \sigma_3$, $\Gamma^2 = 1$, which classifies the states with $\Gamma = +1$ and -1, by convention, as bosonic and fermionic states. Since $[\Gamma, H] = 0$ and $\{\Gamma, Q_a\} = 0$, the Hamiltonian and supercharges are identified as bosonic and fermionic generators. The cases of unbroken and broken supersymmetry are distinguished by the Witten index Δ, defined as the difference between the total numbers of bosonic and fermionic states. In a *nonperiodic* one-dimensional system, unbroken supersymmetry is characterized by one singlet ground state of zero energy and $\Delta \neq 0$; for broken supersymmetry there is no zero energy singlet state and $\Delta = 0$. However, as it was observed in [5,6], supersymmetric *periodic* systems may support two zero energy ground states, and then $\Delta = 0$ in the unbroken case. Some systems investigated there possess a specific property of *self-isospectrality*, meaning that corresponding superpartner potentials V^+ and V^- are given by the same periodic function shifted for a half of a period $2L$, $V^+(x + L) = V^-(x)$. They belong to a class of finite-gap periodic systems, which play an important role, in particular, in condensed matter physics [7] and in the theory of nonlinear integrable systems [8].

In a *nonlinear* generalization of supersymmetric quantum mechanics [9], supercharges Q_a are higher $(n > 1)$ order differential operators generating a nonlinear superalgebra $\{Q_a, Q_b\} = 2\delta_{ab}P_n(H)$, with $P_n(H)$ a polynomial of order n. The number of singlet states can take there any value from 0 to n, and, as in periodic models with linear supersymmetry, the Witten index does not characterize supersymmetry breaking [9,10]. This indicates that in periodic finite-gap systems nonlinear supersymmetry may play an important role.

To investigate the question of the presence and nature of nonlinear supersymmetry in periodic finite-gap systems, in this Letter we study a planar model described by the Pauli Hamiltonian for a nonrelativistic electron in periodic electric and magnetic fields. The model belongs to a broad class of periodic systems investigated by Novikov *et al.* [11]. It is well known that in the absence of an electric field the model, which includes the Landau problem as a particular case, is characterized by a supersymmetry with the usual linear superalgebraic structure (1) [3]. We choose periodic magnetic and electric fields in such a form that the spin-up and -down components of the electron wave function feel the same one-dimensional effective periodic potential but with a shift of half of the period. As a result, the effective potential of superextended system satisfies a property of self-isospectrality. Vector and scalar potentials are fitted to produce the associated Lamé equation with two integer parameters m and l, which belongs to a broad class of finite-gap systems with a smooth potential; see Eq. (3) below [8,12,13]. We find here a special nonlinear supersymmetry of the previously unknown structure, in which all the peculiarities of the band structure of the system are imprinted. In the infinite-period limit our system provides an unusual picture of supersymmetry breaking rooted in its nonlinearity.

Model.—Consider a nonrelativistic electron confined to a plane and moving in the presence of an electric field, given by a scalar potential $\phi(x, y)$, and a perpendicular magnetic field $B_z(x, y)$. It is described by the Pauli Hamiltonian

$$H_e = (p_x + A_x)^2 + (p_y + A_y)^2 + \sigma_3 B_z - \phi, \tag{2}$$

where the units are $\hbar = c = 2m = -e = 1$. Let us restrict B_z and ϕ by the condition that they depend only on x. We choose $A_x = 0$, $A_y = w(x)$, then $B_z = \frac{dw}{dx}$. Present the wave function in the form $\Psi(x, y) = e^{i\kappa y}\psi(x)$, where κ, $-\infty < \kappa < \infty$, is the eigenvalue of p_y. Taking $w(x) = \alpha \frac{d}{dx} \ln(\mathrm{dn}x)$ and $\phi(x) = \beta w^2(x) + \gamma w(x) + \delta$, with appropriate choice of constant parameters α, β, γ and δ, we reduce (2) to a quantum periodic system given by the diagonal matrix Hamiltonian H with up $(+)$ and down $(-)$ components of the form $H^{\pm}_{m,l} = -\frac{d^2}{dx^2} + V^{\pm}_{m,l}(x)$. Here $V^{+}_{m,l}(x) = V^{-}_{m,l}(x + L)$,

$$V^{-}_{m,l}(x) = -C_m \mathrm{dn}^2 x - C_l \frac{k'^2}{\mathrm{dn}^2 x} + c, \tag{3}$$

$C_m = m(m + 1)$, $C_l = l(l + 1)$, m and l are integers such that $C_m^2 + C_l^2 \neq 0$, c is a constant; $\mathrm{dn}x = \mathrm{dn}(x, k)$ is the Jacobi even elliptic function, satisfying a relation $\mathrm{dn}(x + K) = k'/\mathrm{dn}x$, with modulus $0 < k < 1$ and real and imaginary periods $2K$ and $4iK'$, $K(k)$ is the elliptic complete integral of the first kind, $K' = K(k')$, and k' is a complementary modulus, $k'^2 = 1 - k^2$. The Hamiltonian H obtained in this way then describes a pair of two *parity-even* associated Lamé systems, shifted one with respect to the other for the half of the real period $2L$ of the potential, that is equal to $2K$ for $C_m \neq C_l$, and K for $C_m = C_l$. The cases with $C_m C_l = 0$ correspond to the Lamé system [8]. By the Landen transformation [14], the case $C_m = C_l$ is reduced to the case of the Lamé system with same value of parameter C_m but with $C_l = 0$, and with the same imaginary period but the real period divided in two. Taking into account that $C_{-m-1} = C_m$, without loss of generality one can assume that $m > l \geq 0$. Isospectral subsystems $H^{+}_{m,l}$ and $H^{-}_{m,l}$ belong to a class of finite-gap systems with even potential and number of energy gaps in the spectrum equal to m [8,12].

Hidden bosonized supersymmetry.—Consider an n-gap periodic system with *even* Hamiltonian $H = -\frac{d^2}{dx^2} + V(x)$, $V(x + 2L) = V(x) = V(-x)$. Its spectrum $\sigma(H)$ is characterized by the band structure $\sigma(H) = [E_0, E_1] \cup \ldots \cup [E_{2n-2}, E_{2n-1}] \cup [E_{2n}, \infty)$, $E_0 < E_1 < \ldots < E_{2n}$, which consists of n valence bands and a conduction band, separated by energy gaps corresponding to n prohibited bands. The $2n + 1$ singlet band-edge states of definite parity and energies E_j, $j = 0, \ldots, 2n$, are given by periodic or antiperiodic states of periods $2L$ or $4L$. The states in the interior of permitted bands are described by Bloch-Floquet quasiperiodic functions, and every internal energy level is doubly degenerate. The energy doublets are distinguished by the reflection (parity) operator R, $R\psi(x) = \psi(-x)$, that is a nonlocal integral of motion. On the other hand, double degeneration of energy levels is a characteristic feature of a quantum mechanical $N = 2$ supersymmetric system. The presence of $2n + 1$ singlets is an indication of the higher order (≥ 3) nature of the corresponding hidden supersymmetry. Any finite-gap system is characterized by a nontrivial integral of motion A_{2n+1} that is a self-conjugate differential operator of the form $A_{2n+1} = i\frac{d^{2n+1}}{dx^{2n+1}} + \alpha_{2n-1}(x)\frac{d^{2n-1}}{dx^{2n-1}} + \ldots \alpha_0(x)$. The (A_{2n+1}, H) is the Lax pair of the nth order Korteweg-de Vries (KdV) equation, and the condition $[A_{2n+1}, H] = 0$ defines the stationary KdV hierarchy. This pair of commuting operators satisfies identically the relation [8]

$$A^2_{2n+1} = P_{2n+1}(H), \qquad P_{2n+1}(H) = \prod_{j=0}^{2n}(H - E_j), \tag{4}$$

and $2n + 1$ singlet states Ψ_j are the common eigenstates of H and A_{2n+1} of the eigenvalues E_j and 0. The square of the self-conjugate operator A_{2n+1} is positive semidefinite, and (4) implies the described band structure of the spectrum. Integral A_{2n+1} is parity-odd, $\{R, A_{2n+1}\} = 0$. Taking into account that $[R, H] = 0$ and $R^2 = 1$, we see that here the reflection operator plays the same role as the operator σ_3 for $N = 2$ superextended matrix system, and the operator $Z = A_{2n+1}$ can be identified as the supercharge. Define a nonlocal odd operator $\tilde{Z} = iRZ$. Odd supercharges $Q_1 = Z$, $Q_2 = \tilde{Z}$ generate the $N = 2$ nonlinear supersymmetry of order $2n + 1$:

$$\{Q_a, Q_b\} = 2\delta_{ab} P_{2n+1}(H). \tag{5}$$

Since this structure appears in the one-dimensional system without matrix (spin) degrees of freedom, the described nonlinear supersymmetry of order $2n + 1$ for any n-gap periodic system with parity-even Hamiltonian is identified as a hidden *bosonized* supersymmetry [15–17].

Supersymmetry and band structure.—Let us return to our finite-gap self-isospectral system. A vector space spanned by singlet band-edge states of the subsystem $H^{+}_{m,l}$ or $H^{-}_{m,l}$ is divided into two vector subspaces formed by $2L$-periodic and $2L$-antiperiodic (i.e., $4L$-periodic) states. The singlet state Ψ_0 with the lowest energy E_0 is $2L$ periodic, and the singlet state of the other edge of the first valence band, Ψ_1, is antiperiodic. The two edge states Ψ_{2j-1} and Ψ_{2j}, $j = 1, \ldots, m$, separated by an energy gap, have the same period [8,12]. Therefore, the space of periodic singlet states has odd dimension, and the space of antiperiodic states has nonzero even dimension. On these two subspaces of singlet states, two irreducible nonunitary representations of the $\mathrm{sl}(2, \mathbb{R})$ algebra are realized. Namely, according to [12,18], the space of $2m + 1$ singlet states of the associated Lamé system with $m > l$ can be treated as a direct sum of two $\mathrm{sl}(2, \mathbb{R})$-representations of dimensions $m - l$ (spin $j_1 = \frac{1}{2}(m - l - 1)$) and $m + l + 1$ (spin $j_2 = \frac{1}{2}(m + l)$). The period of the states of these subspaces is dictated by the parity of $m - l$. When $m - l$ is odd, spin-j_1 (spin-j_2) representation is realized on $2L$-periodic ($4L$-periodic) states, for even $m - l$ the periodicity of spin-j_1 and spin-j_2 subspaces interchanges. Making use of the two corresponding algebraization schemes [18], we find two commuting antidiagonal self-conjugate integrals of motion, X and Y, where

$$Y = i^{\epsilon_Y}\begin{pmatrix} 0 & Y^-_{m,l}(x) \\ Y^+_{m,l}(x) & 0 \end{pmatrix}, \qquad (6)$$

$Y^-_{m,l}(x) = \frac{\mathrm{dn}^{m+1}x}{\mathrm{cn}^{m+l+2}x}\left(\frac{\mathrm{cn}^2 x}{\mathrm{dn}x}\frac{d}{dx}\right)^{m+l+1}\frac{\mathrm{dn}^l x}{\mathrm{cn}^{m+l}x}$, $Y^+_{m,l}(x) = Y^-_{m,l}(x+K)$, $\epsilon_Y = 1$ (0) for $m+l$ even (odd), and X has a form similar to (6) with $X^-_{m,l}(x) = Y^-_{m,-l-1}(x)$, $X^+_{m,l}(x) = X^-_{m,l}(x+K)$, $\epsilon_X = 1 - \epsilon_Y$. The order of the differential operator X, $|X| = m - l$, is less than the order of Y, $|Y| = |X| + 2l + 1 = m + l + 1$. X and Y have opposite parities $(-1)^{m-l}$ (X), and $(-1)^{m-l+1}$ (Y). When $m - l$ is odd, $2L$-periodic ($4L$-periodic) singlet states of each subsystem are zero modes of the integral X (Y). For even $m - l$ the role of the operators X and Y as annihilators of periodic and antiperiodic edge-state interchanges. The anticommutator $\{X, Y\} = 2Z$ produces a diagonal integral $Z = \mathrm{diag}(Z^+_{m,l}, Z^-_{m,l})$, $|Z| = |X| + |Y| = 2m + 1$, with down,

$$Z^-_{m,l}(x) = iY^+_{m,l}(x)X^-_{m,l}(x) = iX^+_{m,l}(x)Y^-_{m,l}(x), \qquad (7)$$

and up, $Z^+_{m,l}(x) = Z^-_{m,l}(x+K)$, components, which are the parity-odd integrals of motion annihilating all the singlet states of the m-gap subsystems $H^-_{m,l}$ and $H^+_{m,l}$ described above. Hence, $Z^2 = P_Z(H)$, where $P_Z(H)$ is a spectral polynomial (4) of order $2m+1$ with $H = diag(H^+_{m,l}, H^-_{m,l})$. From the explicit form of the integrals X, Y and Z one finds that $[Z, X] = [Z, Y] = 0$, and $X^2 = P_X(H)$, $Y^2 = P_Y(H)$. The polynomials $P_X(H)$ and $P_Y(H)$ factorize the spectral polynomial $P_Z(H)$, $P_Z(H) = P_X(H)P_Y(H)$, and include those factors $(H - E_j)$ for which E_j's are the eigenvalues of band-edge states of corresponding periodicity [19].

Periodic (antiperiodic) states have an even (odd) number of nodes in the period interval. The maximal number of nodes that can have the band-edge states annihilated by $X^\pm_{m,l}$ and $Y^\pm_{m,l}$ is not more than the order of these differential operators. When $m - l$ is *even*, X annihilates $m - l$ antiperiodic band-edge states with $1, 1, 3, 3, \ldots, m-l-1, m-l-1$ nodes. The $m + l + 1$ periodic edge states annihilated by Y have $0, 2, 2 \ldots, (m+l), (m+l)$ nodes. We find that when $m - l$ is *odd*, X annihilates $m - l$ periodic band-edge states with $0, 2, 2, \ldots, m-l-1, m-l-1$ nodes if $m - l > 1$, and one nodeless state Ψ^-_0 if $m - l = 1$. In the last case X is the usual first order supercharge [20]. The operator Y annihilates $m + l + 1$ antiperiodic states with $1, 1, \ldots, m+l, m+l$ nodes.

The picture can be summarized as follows. The band-edge state Ψ_0 is a zero mode of the parity-odd supercharge, i.e., of Y (X) when $m - l$ is even (odd). The band-edge states of the same prohibited band "attract" each other; they appear as zero modes of the same supercharge. When the number of permitted bands $m + 1$ is fixed, and $m - l$ increases in steps of 2, there appear two new band-edge states annihilated by X, with increasing energies. Every such pair is separated by a pair of zero modes of Y. The highest $2(l+1)$ singlet states are zero modes of Y. These properties are illustrated on Fig. 1.

Integral Z reflects the degeneration of the states of each subsystem, while X and Y reveal the self-isospectrality of the composed system. As a result, it is characterized by the fourfold degeneration of quasiperiodic states and the double degeneration of the band-edge states.

Nonlinear superalgebra.—Besides nontrivial integrals X, Y and Z, our system is characterized also by mutually commuting integrals $\Gamma_1 = \sigma_3$, $\Gamma_2 = R$ and $\Gamma_3 = \sigma_3 R$. Any of them can be chosen as the operator Γ that classifies all the integrals into bosonic and fermionic operators. Appropriate linear combinations of physical states for which $\Gamma = +1$ and -1 are identified as bosonic and fermionic states. Let us choose $\Gamma = \sigma_3$. Eight integrals X, $\Gamma_i X$, Y and $\Gamma_i Y$, $i = 1, 2, 3$, anticommuting with Γ are identified as fermionic operators. They anticommute between themselves for certain linear combinations of the 8 bosonic operators Z, $\Gamma_i Z$, Γ_i and H with coefficients that are some polynomials in H. Linear combinations of the bosonic operators $\mathcal{J}^{(\pm)}_1 = -\frac{i}{2}R\sigma_3 Z\Pi_\pm$, $\mathcal{J}^{(\pm)}_2 = \frac{1}{2}\sigma_3 Z\Pi_\pm$ and $\mathcal{J}^{(\pm)}_3 = -\frac{1}{2}R\Pi_\pm$, where $\Pi_\pm = \frac{1}{2}(1 \pm \sigma_3)$, have the only nontrivial commutators

$$[\mathcal{J}^{(\pm)}_a, \mathcal{J}^{(\pm)}_b] = i\rho_c(H)\epsilon_{abc}\mathcal{J}^{(\pm)}_c. \qquad (8)$$

Here a, b, $c = 1, 2, 3$, $\rho_{1,2} = 1$, $\rho_3 = P_Z(H)$. This is a nonlinear deformation of $su(2) \oplus su(2) \oplus u(1) \oplus u(1)$, where the two last terms correspond to $\Gamma = \sigma_3$ and H. The nonlinear algebra (8) is reminiscent of nonlinear symmetry algebra generated by the angular momentum and Laplace-Runge-Lenz vector operators in the quantum Kepler problem [21]. The complete superalgebra is identified as a nonlinear deformation of the $su(2|2)$ superunitary symmetry, in which H plays a role of the multiplicative central charge [22].

Infinite-period limit.—In the self-isospectral system considered here, band-edge states form energy doublets, quasiperiodic states are organized in quadruplets. In the infinite-period limit, corresponding to $k \to 1$, $k' \to 0$, $K \to \infty$, $\mathrm{dn}(x, k) \to \frac{1}{\cosh x}$, the system transforms into a pair of

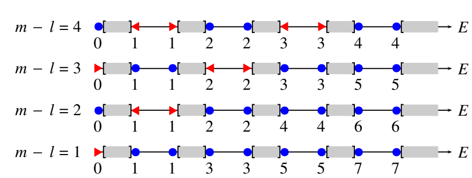

FIG. 1 (color online). Scheme of band structure for self-isospectral systems with $m = 4$. Triangles (dots) indicate band-edge states annihilated by X (Y), the digits below mean their node numbers. The states with even (odd) number of nodes are periodic (antiperiodic).

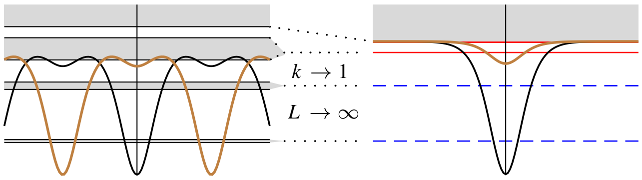

FIG. 2 (color online). Qualitative picture of supersymmetry breaking in a self-isospectral system with $m = 3$, $l = 1$ in the infinite-period limit. The form of potentials and bands shown on the left corresponds to the modulus k close to 1. Two lower horizontal dashed lines on the right show energy levels of singlet bound states, the upper separated horizontal continuous line corresponds to a doublet of bound states, the line at the bottom of continuous spectrum indicates a doublet of the lowest states of the scattering sector.

reflectionless Pöschl-Teller systems given by potentials $V^{\pm}_{m,l} = -C^{\pm}\cosh^{-2}x + c$, $C^+ = C_l$, $C^- = C_m$. In this limit the spectral polynomial degenerates [16]. The quasi-periodic states of the conduction band reduce to the scattering states, supersymmetric doublet of its band-edge reduces to a doublet of the lowest states of the scattering sector. In a shifted subsystem $H^+_{m,l}$, $m - l$ lower valence bands disappear, while the rest of them in both subsystems shrink to the bound states. The resulting system is characterized by $m - l$ singlet and l doublet bound states, and by a doublet of the lowest states of the scattering sector. The rest of the scattering states is organized in energy quadruplets. This unusual picture of supersymmetry breaking, illustrated on Fig. 2, is related to the nonlinear nature of self-isospectral supersymmetry.

Conclusion.—In the model investigated here the nonlinear self-isospectral supersymmetry originates from: (i) the separability of the singlet band-edge states of both subsystems into two nonempty subspaces of periodic and antiperiodic states, and (ii) the related factorization of the higher order Lax pair operator of the associated stationary KdV hierarchy. The mutually commuting integrals X and Y are the annihilators of the band-edge states of definite periodicity. They factorize the integral Z that annihilates all the band-edge states. The unusual nonlinear supersymmetry generated by these nontrivial integrals together with integral σ_3 and parity operator R, reveals the band structure of the system and all its peculiarities in the same way as a the nonlinear symmetry associated with the Laplace-Runge-Lenz vector reflects specific properties of the hydrogen atom spectrum [21].

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