

## Unified Description of $0^+$ States in a Large Class of Nuclear Collective Models

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A remarkably simple regularity in the energies of  $0^+$  states in a broad class of collective models is discussed. A single formula for all  $0^+$  states in flat-bottomed infinite potentials that depends only on the number of dimensions and a simpler expression applicable to all three interacting boson approximation symmetries in the large  $N_B$  limit are presented. Finally, a connection between the energy expression for  $0^+$  states given by the X(5) model and the predictions of the interacting boson approximation near the critical point of the first order phase transition is explored.

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The evolution of structure in many-body quantum systems and the emergence of collective phenomena is a subject that pervades many areas of modern physics. Recently, significant strides have been taken in the study of structural evolution in atomic nuclei, particularly with the discovery of nuclei that undergo quantum phase transitions [1,2] in their equilibrium shapes, and the development of descriptions of nuclei at the phase transitional point by simple, parameter-free critical point symmetries, E(5) [3] and X(5) [4], that invoke flat-bottomed, infinite potentials. These descriptions have been well supported by experimental studies [5–8] and also have application [9] in other systems such as molecules. Thus their study, and that of related models, potentially offers insight into a variety of phase transitional behavior.

E(5) and X(5) are analytic solutions of the Bohr Hamiltonian [10] that describe collective properties in terms of two shape variables—the ellipsoidal deformation  $\beta$  and a measure of axial asymmetry,  $\gamma$ . Both use an infinite square well in  $\beta$ , differing in their  $\gamma$  dependence. Their success has given rise to numerous other geometrical models, many of which can also be solved analytically. Examples are X(3) [11] in which the  $\gamma$  potential is frozen at  $0^\circ$ , Z(4) [12], in which it is fixed at  $\gamma = 30^\circ$ , and Z(5) [13] which has a minimum at  $\gamma = 30^\circ$ . In all these models, the number in parentheses is the effective dimensionality,  $D$ . For example, the 5-dimensional models are couched in terms of  $\beta$ ,  $\gamma$  and the three Euler angles.

The nature of low lying  $0^+$  states is critical to understanding the structure of nuclei. Their identification and interpretation is a subject of recent experimental [14,15] and theoretical work, both in the microscopic quasiparticle phonon model [16] and in the relativistic mean field framework [17].

It is the purpose of this Letter to show that a large class of seemingly diverse models in fact share some remarkable similarities and to exemplify this by pointing out certain heretofore unrecognized but simple and general regularities in the energies of  $0^+$  states which characterize these models. We will obtain a single, simple formula for *all*  $0^+$

states in any flat-bottomed infinite potential that depends *solely* on the number of dimensions and another even simpler expression applicable to the dynamical symmetries of the interacting boson approximation (IBA) model [18] (in the limit of large valence boson number,  $N_B$ ) and compare these results to available data. We will also use solutions for a series of potentials with intermediate shapes to study the evolution from one description (formula) to the other. Finally, we will show that IBA predictions near the critical point, for large  $N_B$ , approach the same energy expression for  $0^+$  states as given by the X(5) critical point symmetry.

In Fig. 1, the IBA triangle is shown with its dynamical symmetries and division into spherical and deformed regimes, separated by a phase transitional region [1]. We include the Alhassid-Whelan arc of regularity [19] and the geometric critical point models E(5) and X(5), although the latter do not belong to the IBA space.

For any infinite flat-bottomed (i.e., square well) potential, the energy eigenvalues are proportional to the squares of roots of the Bessel functions  $J_\nu(z)$  where the order  $\nu$  is different for each case. In E(5) [3], one has

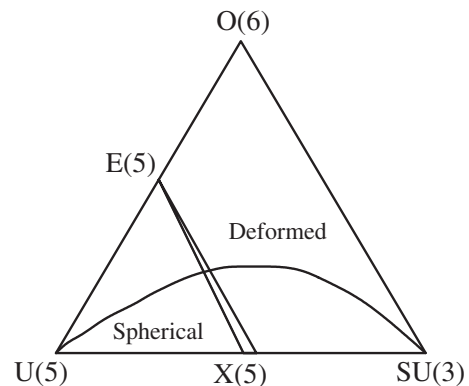


FIG. 1. IBA symmetry triangle with the three dynamical symmetries. The critical point models E(5) and X(5) are placed close to the phase transition region (slanted lines). The solid curve indicates the Alhassid-Whelan arc of regularity.

$$\nu = \tau + 3/2, \quad (1)$$

with  $\tau = L/2$ . In X(5) one has [4]

$$\nu = \sqrt{\frac{L(L+1)}{3} + \frac{9}{4}} \quad (2)$$

for all  $K = 0$  bands. In Z(5) one has [13]

$$\nu = \frac{\sqrt{L(L+4) + 3n_w(2L - n_w) + 9}}{2}, \quad (3)$$

with  $n_w = 0$  for all  $0^+$  bandheads. As a result, in all three cases, the  $0^+$  bandhead energies are given by the squares of subsequent roots of  $J_{3/2}(z)$ .

These energies are given in Table I and are plotted on the left in Fig. 2. While these models seem quite different, if we normalize each energy to that of the first excited  $0^+$  state—that is, consider relative  $0^+$  eigenvalues—we see that these different models produce exactly identical results as seen on the right in Fig. 2 and given by (see column Norm in Table I): 0, 1, 2.5, 4.5, 7, 10. These energies are approximately described by the simple formula

$$E = An(n+3) \quad (4)$$

where  $n$  is the ordinal number of the  $0^+$  state and where  $A$  depends on the model.

This remarkable result systematizes the  $0^+$  states of this wide array of seemingly diverse models. However, its significance runs much deeper. Consider now the models Z(4) in four dimensions and X(3) in three.

In Z(4) the order of the Bessel functions is given by

$$\nu = \frac{\sqrt{L(L+4) + 3n_w(2L - n_w) + 4}}{2}, \quad (5)$$

with  $n_w = 0$  for all  $0^+$  states, giving  $J_1(z)$  for  $L = 0$ .

In X(3) one has [11]

$$\nu = \sqrt{\frac{L(L+1)}{3} + \frac{1}{4}} \quad (6)$$

leading to  $J_{1/2}(z)$  for  $L = 0$ .

TABLE I. Energies of  $0^+$  states in the E(5), Z(5), and X(5) models. Energies on the left are in units of  $E(2_1^+) = 1.0$ , while in the column Norm, in units  $E(0_2^+) = 1.0$ . The normalized results are identical for each of the models. The column IBA-Norm gives the normalized  $0^+$  energies for a large  $N_B$  IBA calculation near the critical point (see text).

$0_i^+$	E(5)	Z(5)	X(5)	Norm	IBA-Norm
$0_1^+$	0	0	0	0	0
$0_2^+$	3.03	3.91	5.65	1.0	1.0
$0_3^+$	7.58	9.78	14.12	2.50	2.48
$0_4^+$	13.64	17.61	25.41	4.50	4.62
$0_5^+$	21.22	27.39	39.53	7.00	7.13
$0_6^+$	30.31	39.12	56.47	10.00	9.85

The  $0^+$  energies given by Z(4) and X(3) can again be described by simple formulas, satisfying

$$E = An(n+2.5) \text{ Z(4)}, \quad E = An(n+2) \text{ X(3)}. \quad (7)$$

The results of Eqs. (4) and (7) exemplify a universal formula that depends solely on the number of dimensions. The energies of all  $0^+$  states in any flat-bottomed infinite square well potential in  $D$  dimensions are given by

$$E = An\left(n + \frac{D+1}{2}\right) \quad (8)$$

where, again,  $A$  depends on the model. Figure 3(a) shows the results of Eq. (8) for  $D = 3, 4$ , and 5. These results reflect the deep relation between the order of the Bessel function solutions and the dimensionality of the potential, given by  $\nu = (D-2)/2$ . Equation (8) is exact for  $D = 3$  and an excellent approximation for low  $\nu$  otherwise. These findings are applicable well beyond the models discussed above. For example, Eq. (8) gives all the  $0^+$  energies in a recent model [20] of the critical point of a pairing vibration to pairing rotation phase transition in which the  $0^+$  states span 2 degrees of freedom—excitation energies within a given nucleus and the sequences of masses along a series of even-even nuclei. Hadronic spectra have also been described [21] by roots of Bessel functions.

The regularities found for  $0^+$  states in solutions of the Bohr Hamiltonian with an infinite well potential in  $\beta$  can be related to the second order Casimir operator of  $E(D)$ , the Euclidean algebra in  $D$  dimensions, which is the semi-direct sum [22] of the algebra  $T_D$  of translations in  $D$  dimensions (generated by the momenta), and the  $SO(D)$  algebra of rotations in  $D$  dimensions (generated by the angular momenta) symbolically written as  $E(D) = T_D \oplus SO(D)$  [23]. The square of the total momentum,  $P^2$ , is a second order Casimir operator of the algebra, and eigenfunctions of this operator satisfy [12]

$$\left(-\frac{1}{r^{D-1}} \frac{\partial}{\partial r} r^{D-1} \frac{\partial}{\partial r} + \frac{\omega(\omega + D - 2)}{r^2}\right) F(r) = k^2 F(r), \quad (9)$$

in the left-hand side of which the eigenvalues of the Casimir operator of  $SO(D)$ ,  $\omega(\omega + D - 2)$  appear [24].

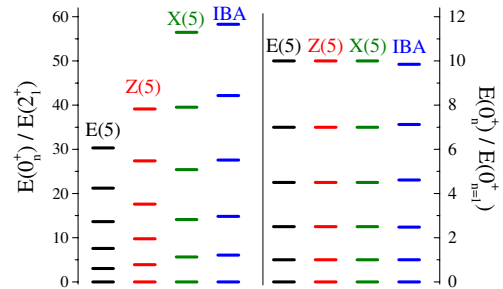


FIG. 2 (color online). (Left) Energies of excited  $0^+$  states in the E(5), Z(5), and X(5) models as well as an IBA calculation near the critical point (see text). (Right) Same as left with the energies normalized to the first excited  $0^+$  state energy.

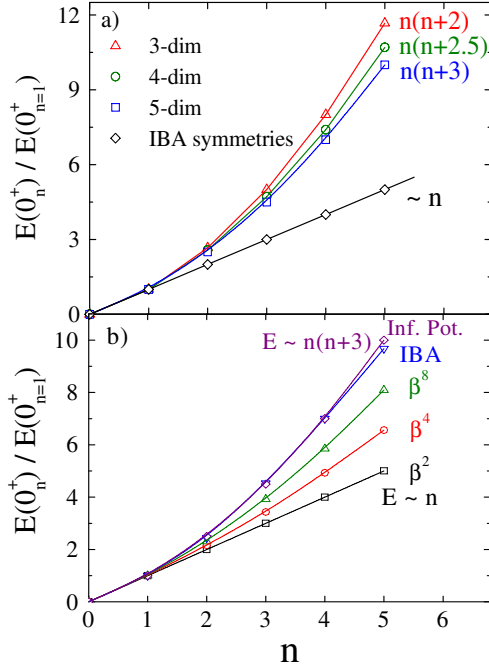


FIG. 3 (color online). Energies of excited  $0^+$  states (normalized to the first excited  $0^+$  state) as a function of their ordinal number. (a) For infinite square well potentials in 3, 4, and 5 dimensions and for the three IBA dynamical symmetries in the large  $N_B$  limit. (b) For potentials of the form  $\beta^{2m}$  as well as for the 5D square well potential and an IBA calculation near the critical point (see text).

Performing the transformation  $F(r) = r^{(2-D)/2} f(r)$ , and using  $\nu = \omega + \frac{D-2}{2}$ , Eq. (9) is brought into the form

$$\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + k^2 - \frac{\nu^2}{r^2} \right) f(r) = 0, \quad (10)$$

whose solutions are Bessel functions  $f(r) = J_\nu(kr)$ .

In the original developments of the infinite square well models, the “radial” equations are obtained, after the above transformation has been performed, in the form of Eq. (10) with the corresponding order  $\nu$ . In E(5) all states obey Eq. (10) (with  $\omega = \tau$ ), in Z(4) agreement occurs for all with  $n_w = 0$  and  $L = 2\omega$  [12], while in X(5), X(3), and Z(5) (with  $n_w = 0$ ) agreement is limited to states with  $L = 0 = \omega$ , i.e., to the  $0^+$  bandheads. This situation resembles a partial dynamical symmetry [25] of type I [26], in which some of the states (here, the  $0^+$  states) preserve all the relevant symmetry.

Do we find these patterns in real nuclei? Table II (left) gives the  $0_3^+$  energies predicted by Eqs. (4) and (8) in comparison to the X(5)-like nuclei  $^{150}\text{Nd}$ ,  $^{152}\text{Sm}$ , and  $^{154}\text{Gd}$ . In each nucleus, there is indeed a  $0^+$  state within  $<100$  keV of the predictions. In  $^{150}\text{Nd}$  this is the  $0_3^+$  state, while in the rest, it corresponds to a higher-lying  $0^+$  state, for which the collectivity still needs to be vetted experimentally through improved spectroscopic information.

Equations (4) and (7) are peculiar to infinite square wells and thus limited to a select number of nuclei. What behav-

TABLE II. Experimental  $0^+$  levels (in keV) of several nuclei compared to the  $0_3^+$  predictions (normalized to the experimental  $0_2^+$  energy) of X(5) and SU(3). For SU(3), states belonging to the  $(2N - 8, 4)$  and  $(2N - 6, 0)$  irreps are given.

	Exp. $0^+$	X(5) $0_3^+$		Exp. $0^+$	SU(3) $0_3^+$
$^{150}\text{Nd}$	1738	1688	$^{156}\text{Gd}$	1851	1916
$^{152}\text{Sm}$	1659	1712		1989	2054
$^{154}\text{Gd}$	1650	1702	$^{158}\text{Gd}$	2276	2201
				2338	2344

ior then characterizes other potentials? Of course, this cannot be solved in general but there is at least one other class of models where an easy solution can be derived, namely, in the dynamical symmetries of the IBA.

In U(5), the  $0^+$  energies are simply proportional to the number of  $d$  bosons defining their respective phonon number; thus,  $E(0_n^+) = An$ . In SU(3), the eigenvalue expression for  $L = 0^+$  states, in terms of the usual representation labels  $(\lambda, \mu)$ , is  $E = a[\lambda^2 + \mu^2 + \lambda\mu + 3(\lambda + \mu)]$ . In O(6), the corresponding equation, in terms of the major family quantum number  $\sigma$ , is (for  $\tau = 0, L = 0$  states)  $E = a\sigma(\sigma + 4)$ . The irreducible representations for SU(3) and O(6) and the corresponding  $0^+$  energies are given in Table III. Taking successive  $(\lambda, \mu)$  and  $\sigma$  quantum numbers and the limit  $N_B \rightarrow \infty$  we obtain

$$E = An. \quad (11)$$

Thus, perhaps surprisingly, considering how different their structures are, and analogous to the infinite flat potentials, a single, simple formula applies to all three dynamical symmetries of the IBA which exhibit identical relative energy spectra of  $0^+$  states in the large  $N_B$  limit. Equations (8) and (11) are compared in Fig. 3(a).

The description of  $0^+$  states with a simple analytic formula extends beyond just the IBA symmetry limits. The behavior of  $0^+$  states in the IBA symmetries can be associated with the chaotic properties of the IBA [19]. A regular region connects U(5) and O(6), resulting from the underlying O(5) symmetry. The regular behavior associated with U(5) and O(6) is preserved along the U(5)-O(6) leg, manifesting quasidynamical symmetries [27], until close to the point of the second order transition. It has been conjectured [28] that the arc of regularity shown in

TABLE III. Irreducible representations (irreps) of SU(3) and O(6) and the corresponding energy of the excited  $0^+$  states.

SU(3)		O(6)	
Irrep $(\lambda, \mu)$	$E(0^+)$	Irrep $(\sigma)$	$E(0^+)$
$(2N, 0)$	0	$(N)$	0
$(2N - 4, 2)$	1	$(N - 2)$	1
$(2N - 8, 4)$	$(4N - 6)/(2N - 1)$	$(N - 4)$	2
$(2N - 6, 0)$	$(4N - 3)/(2N - 1)$	$(N - 6)$	$3 - (3/N)$
$(2N - 12, 6)$	$(6N - 15)/(2N - 1)$	$(N - 8)$	$4 - (8/N)$

Fig. 1—a region of ordered behavior within the triangle—is related to an underlying partial SU(3) symmetry (up to the critical line). Table II (right) shows good agreement of the SU(3) predictions for the nearly degenerate  $0^+$  companion states from Table III with two well-deformed nuclei proposed [29] to lie along the arc, although the collectivity of these  $0^+$  states needs further experimental examination.

Most nuclei, however, are not described by dynamical symmetries or critical point models but lie somewhere in between [30]. To study how Eqs. (4) and (11) evolve into one another in the symmetry triangle, we now consider a sequence of potentials of the form  $V \sim \beta^{2m}$  starting from  $\beta^2$ , corresponding to U(5), and ending at either X(5) or E(5), which is successively approached by increasing powers of  $\beta$ . The results are shown in Fig. 3(b). As the potential flattens with increasing powers of  $\beta$ , the results go from those for the U(5) limit to those for the infinite square well potentials. In each case, the normalized  $0^+$  energies are well reproduced by a formula analogous to Eq. (4), namely,  $E \sim n(n+x)$  where  $x \rightarrow \infty$  for the IBA symmetries and drops to 3 for E(5)/X(5).

Finally, it has recently been discussed how the IBA with appropriate parameters approaches the predictions of the critical point symmetries as  $N_B \rightarrow \infty$  [31]. We use an IBA Hamiltonian in the form [32]

$$H(\zeta, \chi) = c \left[ (1 - \zeta) \hat{n}_d - \frac{\zeta}{4N_B} \hat{Q}^\chi \cdot \hat{Q}^\chi \right], \quad (12)$$

where  $\hat{n}_d = d^\dagger \cdot \tilde{d}$ ,  $\hat{Q}^\chi = (s^\dagger \tilde{d} + d^\dagger s) + \chi (d^\dagger \tilde{d})^2$ ,  $N_B$  is the number of valence bosons, and  $c$  is a scaling factor. Calculations were performed with the IBAR code [33].

Included in Fig. 3(b) is an IBA calculation with  $\chi = -\sqrt{7}/2$  (bottom leg of the triangle),  $N_B = 250$ , and  $\zeta = 0.473$  [31], which is very close to the critical point ( $\zeta_{\text{crit}} = 0.472$ ) of the phase transition region in the IBA. One sees that the IBA results are very close to those of Eq. (4) consistent with the flat nature of the IBA energy surface near the critical point. This same result is also illustrated in Fig. 2. The normalized IBA energies are included in Table I and show a strong similarity with the results of the infinite square well potentials. Thus, it appears that the regularities in  $0^+$  energies obtained in the infinite square well potentials are not restricted to geometrical models and also occur near the critical point of the IBA.

In summary, we have discussed very simple regularities in excited  $0^+$  energies which pervade a number of different models. The energies of excited  $0^+$  states in flat-bottomed infinite potentials can be described by a single expression dependent on only the number of dimensions. These observed regularities in  $0^+$  energies are linked to the second order Casimir operator of  $E(n)$ . Further, the energies of  $0^+$  states in all three dynamical symmetries of the IBA are governed, in the large  $N_B$  limit, by a single expression. These results were compared to experimental data. Potentials with shapes intermediate between the U(5) symmetry and the infinite square well give a smooth evolution

in the  $0^+$  energies. Finally, IBA calculations with large  $N_B$ , near the critical point of the phase transition, exhibit nearly the same energy dependence for  $0^+$  states as given by the infinite square well potentials.

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