Finite-Layer Thickness Stabilizes the Pfaffian State for the 5/2 Fractional Quantum Hall Effect: Wave Function Overlap and Topological Degeneracy

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We find the finite width, i.e., the layer thickness, of experimental quasi-two-dimensional systems produces a physical environment sufficient to stabilize the Moore-Read Pfaffian state thought to describe the fractional quantum Hall effect at filling factor $\nu = 5/2$. This conclusion is based on exact calculations performed in the spherical and torus geometries, studying wave function overlap and ground state degeneracy.

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Introduction.—Two-dimensional (2D) electrons strongly interacting in the presence of a perpendicular magnetic field experience the fractional quantum Hall effect [1] (FQHE) at certain fractional electronic Landau level (LL) filling factors ν characterized by an incompressible state with fractionally charged quasiparticles with anyonic, rather than fermionic, statistics, the observation of which requires clean (high mobility) samples, low temperatures, and high magnetic fields. The FQHE abounds in the lowest Landau level (LLL) with the observation of over 70 odd denominator FQHE states, the most famous being the Laughlin state [2] describing the FQHE at $\nu = 1/m$ (m odd)—the odd denominator a consequence of the Pauli exclusion principle. We are concerned with the FQHE in the second LL (SLL) where the FQHE is scarce with only about 8 observed FQHE states which tend to be fragile with low activation energies.

The most discussed FQH state in the SLL occurs at the even-denominator filling factor $\nu = 5/2$ [3], thought to be described by the Moore-Read Pfaffian [4] state (Pf) which intriguingly possesses quasiparticle excitations with non-Abelian statistics providing the tantalizing possibility of topological quantum computation [5]. The presence of this state challenges our understanding and suggests the condensation of bosons (perhaps fermion pairs) in a new type of incompressible fluid. Although the Pf state is the leading candidate for the observed 5/2 FQHE, the *actual* nature of the state is currently debated [6–8]. Considering the importance of this state, our apparent lack of understanding of its precise nature, more than 20 years after its discovery, is both embarrassing and problematic. This is particularly true in view of the existence (for more than 15 years) of a beautiful candidate 5/2 FQHE state, viz., the Pf state [4].

The Pf is not as successful in describing the FQHE at $\nu = 5/2$ as the Laughlin theory is in describing the FQHE in the LLL indicated by the modest overlap between the Pf wave function and the exact Coulomb Hamiltonian wave function [9,10] (approximately ~ 0.9 compared to ~ 0.999 for the Laughlin theory wave functions). However, changing [10] the short range components of the Coulomb interaction can produce an exact wave function with near unity overlap with the Pf. Furthermore, the actual electron-electron interaction in the FQHE experimental systems is not purely Coulombic due to additional physical effects such as disorder, LL mixing, finite-thickness due to the quasi-2D nature of the system, etc. A natural question arises: can any of these effects be incorporated to produce an exact state that is accurately described by the Pf wave function? We answer this question affirmatively with one of the simplest extensions of the pure Coulombic interaction, namely, the inclusion of finite-thickness effects.

We find, by including the finite-thickness effects perpendicular to the 2D plane, the exact ground state is very successfully approximated by the Pf model. We consider two different complementary compact geometries—the sphere [11] and torus [12]. Throughout this work we assume the electrons exactly fill half of the SLL yielding an electron filling factor of $\nu = 2 + 1/2 = 5/2$ (two coming from completely filling the lowest spin-up and -down bands). Furthermore, we assume electrons in the SLL to be spin-polarized since the current consensus supports that conclusion (in any case the Pf describes a spin-polarized state) and ignore disorder or LL mixing effects (neglecting LL mixing effects may not be a very good approximation for the 5/2 FQHE [13]). Hence, the Hamiltonian is merely the spin-polarized electron interaction Hamiltonian.

Haldane [11] showed the Hamiltonian, of interacting electrons confined in the SLL, can be parameterized by pseudopotentials $V_m^{(1)}$—the interaction energies between any pair of electrons with relative angular momentum m

$$V_m^{(1)} = \int_0^\infty dkk[L_1(k^2/2)]^2 L_m(k^2) e^{-k^2} V(k), \qquad (1)$$

with $V(k)$ the Fourier transform of the interaction potential and $L_n(x)$ Laguerre polynomials. We model the quasi-2D nature of the experimental system (finite-thickness) by an infinite square-well potential in the direction perpendicular to the electron plane, since the best experimental system for the observation of the 5/2 FQHE is typically the GaAs quantum well structure well described by this model (discussed elsewhere [14]), given by

$$V(k) = \frac{e^2 l}{\epsilon}\frac{1}{k}\frac{(3kd + \frac{8\pi^2}{kd} - \frac{32\pi^4[1-\exp(-kd)]}{(kd)^2[(kd)^2+4\pi^2]})}{(kd)^2 + 4\pi^2}, \qquad (2)$$

where ϵ is the dielectric constant of the host semiconductor and $l = \sqrt{\hbar c/eB}$ is the magnetic length. Equation (1) applies to the planar geometry; hence, for the torus it is exact; however, we also use it on the sphere since (i) it can be argued it better represents the thermodynamic limit, and (ii) convenience. We do not expect any qualitative error arising from using these pseudopotentials for spherical system diagonalization.

Spherical geometry.—This geometry consists of N_e electrons confined to the spherical surface with a radial magnetic field produced by a magnetic monopole of strength $N_\phi/2$ at the center yielding a total magnetic flux piercing the surface of N_ϕ (N_ϕ is an integer due to Dirac's quantization condition). The total angular momentum L is a good quantum number and incompressible states are uniform states with $L = 0$ and a nonzero energy gap. The filling factor is $\nu = \lim_{N_e\to\infty} N_e/N_\phi$. Using Eq. (1) we calculate entirely within the LLL. The relationship between N_e and N_ϕ for the Pf state, describing filling 1/2 in the SLL, is $N_\phi = 2N_e - 3$ with -3 known as the "shift."

An appropriate measure to determine the accuracy of the Pf description of the 5/2 FQHE is the overlap between the exact ground state and the variational Pf wave functions. An overlap of unity (zero) indicates the two states are completely alike (different). Overlap calculations have been influential in establishing the nature of the FQHE in the LLL—in particular, the primary reason for the theoretical acceptance of the Laughlin wave function as the appropriate description for the 1/3 FQHE is the large (>99%) overlap it has with exact finite size numerical many-body wave functions. In the upper, middle, and lower panels of Fig. 1 we show the overlap between the exact ground state for some finite-thickness value d (= quantum well width) and the Pf wave function for $N_e = 8$, 10, and 12 electrons, respectively, (note that the $N_e = 12$ system is *aliased* with a FQH state at filling 2/3 and its identification with 1/2 is dubious [9]). In the zero width case the overlap is relatively modest but encouraging (~0.9). Surprisingly, the inclusion of finite width causes the overlap to *increase* to a maximum before inevitably decreasing to zero for large d. Furthermore, the maximum occurs at nearly the same value of $d = d_0 \sim 4l$ for different system sizes indicating this conclusion survives in the thermodynamic limit. Work by the authors [14] showed this effect is true for other models of finite thickness with a similar d_0. Thus, the increase of the Pf overlap with the well width is a generic qualitative phenomenon, independent of the finite-thickness model employed. It appears that weakening the Coulomb coupling by increasing d (to about $4l$) creates an interaction Hamiltonian favorable to the Pf description. We mention (emphasized in Ref. [14]) that increasing overlap with increasing layer thickness does *not* happen at all for the LLL FQHE where it is known [15] that increasing layer thickness strongly suppresses the overlap, leading eventually to the destruction of the FQHE—e.g., the Laughlin $1/m$ overlap is always maximum at $d = 0$.

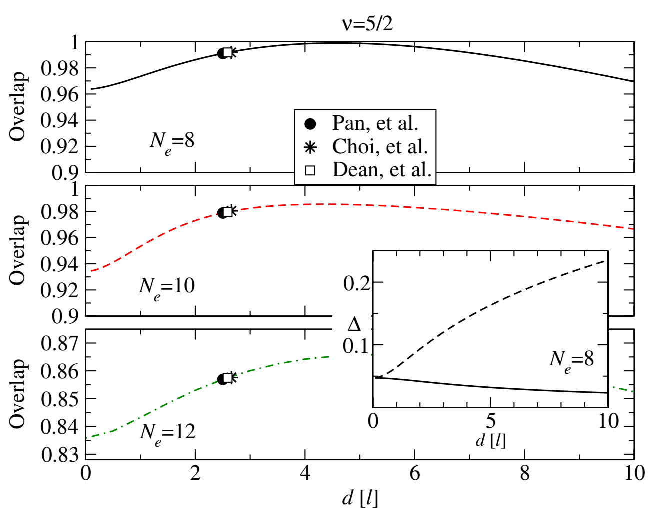

FIG. 1 (color online). Wave function overlap between the exact ground state, for a quasi-2D system modeled by an infinite square-well potential, and the Pf as a function of finite width d for $N_e = 8$, 10, and 12 electrons, respectively. The experimental d of Refs. [24] (solid circle), [25] (star), and [13] (open square) are also indicated. The inset shows the excitation gap Δ for $N_e = 8$ as a function of d both in units of $e^2/\epsilon l$ (solid) and $e^2/\epsilon\sqrt{l^2 + d^2}$ (dashed).

Torus geometry.—To test the robustness of this conclusion we study the Pf state on the torus. These results are, in a sense, our main results because they (i) have less system-size dependence, and (ii) are more general, i.e., independent of the detailed form of the Pf wave function and dependent only on the topological nature of the underlying 5/2 ground state. In fact, the finite d spherical geometry results serve as our motivation and inspiration to investigate the ground state topological degeneracy on the torus at finite d, finding the remarkable topological degeneracy—the hallmark of a non-Abelian state.

On the torus, there is no shift in the relation between N_e and N_ϕ making a direct comparison possible between various quantum phases at a given filling factor, such as a Pf state, composite fermion (CF) Fermi sea, or a stripe phase, all possible at $\nu = 1/2$ in the SLL. These competing states have different spectral signatures identified by using periodic rectangular geometry with sides a and b. The magnetic field prevents standard translation operators from commuting; however, Haldane [12] showed one can construct many-body eigenstates with two conserved pseudomomenta associated with the translations. The two-dimensional pseudomomentum $\mathbf{K}$ exist in a Brillouin zone containing only N_0^2 points where N_0 is the greatest common divisor of N_e and N_ϕ and K_x (K_y) are in units of $2\pi\hbar/a$ ($2\pi\hbar/b$). There is an exact (trivial) degeneracy q

due to the center of mass motion at filling factor p/q which we ignore as it is unrelated to the physics (independent of the Hamiltonian). In the rectangular unit cell the discrete symmetries relate states at $(\pm K_x, \pm K_y)$ and as a consequence, we only consider states with pseudomomenta in the range $(0 \ldots N_0/2, 0 \ldots N_0/2)$.

To render the Laughlin state periodic [16] the recipe is to replace the Jastrow factor $(z_i - z_j)^m$ by a Jacobi theta function $\theta_1(z_i - z_j|\tau)^m$ where $\tau = ib/a$ with z the complex electron position. The Jacobi theta function has the quasiperiodicity required to construct a Laughlin state at $\nu = 1/m$ observed in numerical studies [16]. This recipe fails when applied to the Pf since there are denominators of the form $(z_i - z_j)$ present and the quasiperiodicity of the theta function does not appear as an overall factor [17]. The correct substitution [18] is

$$1/(z_i - z_j) \rightarrow \theta_a(z_i - z_j|\tau)/\theta_1(z_i - z_j|\tau), \qquad a = 2, 3, 4 \tag{3}$$

leading to *three* candidate ground states. This degeneracy is topological in origin and a signature of the special properties of the Pf state. To our knowledge, no earlier work in the literature has directly discovered this topological degeneracy on the torus for the 5/2 state in spite of its great significance. In the Pf phase of the real system, such as electrons interacting via Coulomb, we expect the degeneracy should be approximate for finite size systems and should become clearer with increasing system size. Note this trend is opposite to the overlap trend shown in Fig. 1 where the overlap decreases slowly with increasing system size (i.e., from $N_e = 8$ to 12) since the Pf is a variational approximation. The wave function of the Pf, when written on the torus using Eq. (3), has pseudomomenta that are half reciprocal lattice vectors. For electrons at $\nu = 1/2$ these states have $\mathbf{K} = (0, N_0/2)$, $(N_0/2, 0)$, $(N_0/2, N_0/2)$. To explicitly separate these states we consider the rectangular unit cell (a hexagonal unit cell has discrete symmetries that render all corners of the magnetic Brillouin zone equivalent resulting in a trivial Pf degeneracy).

We have performed exact diagonalizations for $N_e = 10$, 12, and 14 electrons at half filling in the SLL. Using a pure Coulomb potential ($d = 0$) we find, for all system sizes, that the spectra are very sensitive to the unit cell aspect ratio and N_e consistent with previous evidence [19] for a nearby compressible stripe phase. The CF Fermi sea also displays sensitivity to boundary conditions and changes of the ground state vector $\mathbf{K}$ with N_e. This is what we observe for the same systems with the LLL Coulomb potential at zero width. No obvious ground state degeneracy can be discerned in our $d = 0$ results.

Switching to nonzero width (using the SQ potential) we find the appearance of a threefold quasidegenerate set of states with the right Pf predicted quantum numbers. This phenomenon is best seen in the region of maximum overlap found in the spherical geometry, i.e., $d = d_0 \sim 4l$; see Fig. 2. In this regime, the spectra are much less sensitive to the aspect ratio, moreover, this behavior is observed for all finite width models. This is the first time the Pf degeneracy has been observed in an electronic topological system, i.e., a system described by a two-body electron-electron interaction Hamiltonian. For bosons at $\nu = 1$ with delta function interactions the degeneracy appears at $d = 0$ [20].

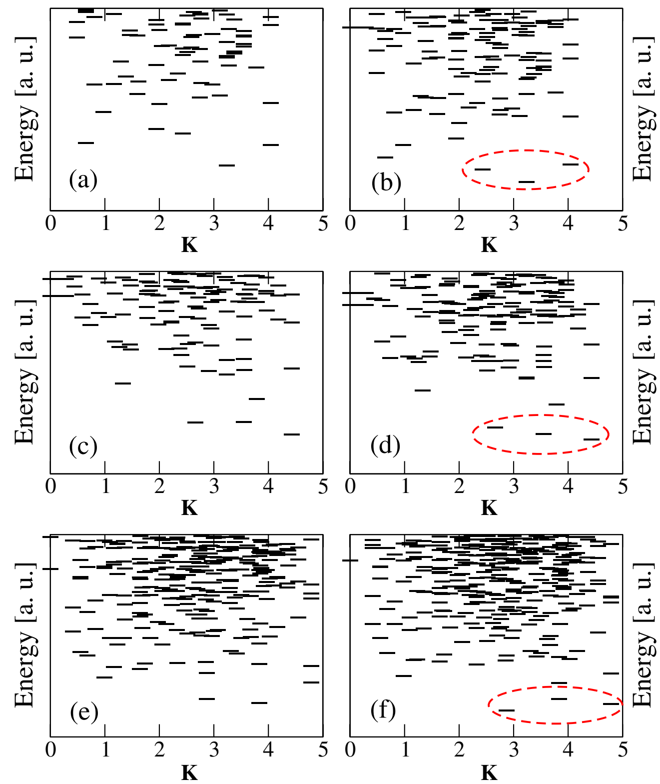

FIG. 2 (color online). Low-lying eigenenergies as a function of the pseudomomentum $\sqrt{K_x^2 + K_y^2}$ (in physical units) for an aspect ratio of $a/b = 0.75$. The left panels refer to zero width while the right panels correspond to the SQ potential of width $d = 4l$ for (a)–(b) $N_e = 10$, (c)–(d) $N_e = 12$, (e)–(f) $N_e = 14$.

Another feature pointing to the appearance of the Pf is related to the role of the particle-hole $(p - h)$ symmetry. The Pf wave function is not invariant under this symmetry [21,22] and its $p - h$ conjugate has been termed the anti-Pfaffian. While the filling factor $\nu = 1/2$ is $p - h$ invariant, the consequences depend upon the geometry. On the sphere, while the Coulomb two-body Hamiltonian has exact $p - h$ symmetry, the Pf wave function has a nontrivial shift -3 implying that its $p - h$ conjugate requires a different flux $N_\phi = 2N_e + 1$. The zero shift on the torus leads to the coexistence of these two states, each having exactly the same threefold topological degeneracy with the same quantum numbers. In a finite system there is no spontaneous breaking of a discrete symmetry and we expect tunneling to lead to symmetric and antisymmetric combinations as eigenstates. We thus expect a doubling of the Pf states due to the $p - h$ symmetry. This is best observed (cf. Fig. 3) on the torus at finite width with a nearly square unit cell where the two states at $(0, N_0/2)$ and

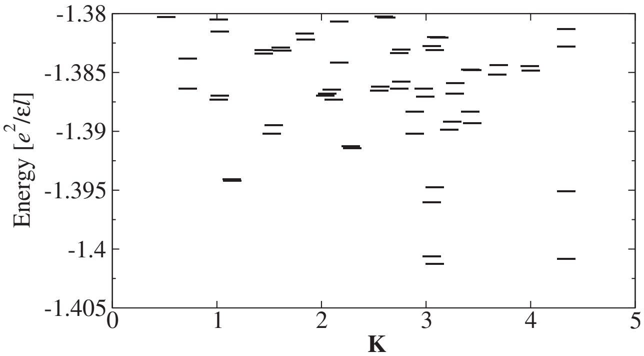

FIG. 3. Low-lying eigenenergies of $N_e = 12$ electrons as a function $\mathbf{K}$ using the SQ potential of width $d = 4l$ and aspect ratio $a/b = 0.99$. There are doublets at the wave vectors of the Pf ground states corresponding to the Pfaffian-anti-Pfaffian symmetric and antisymmetric combinations. Because of the almost square symmetry, there are quasidegeneracies between states at $\mathbf{K} = (6, 0)$ and $\mathbf{K} = (0, 6)$.

$(N_0/2, 0)$ (exactly degenerate for the square unit cell) are very close in energy and all three members of the Pf multiplet have exactly one partner at a slightly higher energy. This does not happen at zero width ($d = 0$) and is strong evidence for the stabilization of the Pf physics in the SLL by finite width effects.

The observation of the topological degeneracy on the torus *only* for finite thickness, $d \sim 4l$, precisely where the overlap is also a maximum on the sphere is, in our opinion, compelling evidence that the 5/2 FQHE is likely to be a non-Abelian state.

Conclusion.—Our results show the—often assumed trivial—effects of the quasi-2D nature of the experimental system produce an exact state *better* described by the Pf. The fact that this conclusion is reached in different finite sized systems for two different geometries (for several models of thickness) is compelling. Our results are not inconsistent with previous work [6–8] in the $d = 0$ limit showing the absence of the Pf. Further, since we find a robust Pf at finite d the transport gap, seen experimentally, would be weaker than predicted in $d = 0$ theoretical studies since finite width "trivially" reduces energy gaps; see the inset of Fig. 1. Thus, the supposed fragility of the 5/2 state may not necessarily be due to it being close to a phase boundary, perhaps between a CF Fermi sea and stripe phase, instead, it may come from the relatively wide quasi-2D system needed to produce a stable Pf.

In this context, it is useful to mention that although earlier theoretical work [9,10] pointed to the importance of tuning the pseudopotential ratio $V_1^{(1)}/V_3^{(1)}$ in stabilizing the Pf, finite width affects [14] *all* pseudopotentials, not just $V_1^{(1)}/V_3^{(1)}$. Tuning $V_1^{(1)}$ and/or $V_3^{(1)}$, while theoretically convenient [9,10], is an ambiguous technique for understanding the stability in real quasi-2D systems where pseudopotentials cannot be tuned arbitrarily. Therefore, our work establishing the optimal stability of the 5/2 Pf at the relatively large width of $d \sim 4l$, is important in view of the fact that the Pf is an exact eigenstate only of a three-body interaction Hamiltonian not expressible in terms of pseudopotentials. As shown in Fig. 1, the current quasi-2D samples typically have $d \sim 2.5l$ where the wave function overlap is large, yet not optimal, as it would be for thicker samples with $d \sim 4l$.

Our direct numerical finding of the appropriate topological degeneracy of the 5/2 FQHE state on the torus and the recent experimental demonstration [23] of the expected $e/4$ quasiparticle charge in shot noise measurements at $\nu = 5/2$, taken together, provide convincing necessary and sufficient conditions supporting the contention that the 5/2 FQHE state is indeed the Moore-Read Pfaffian wave function (or some other equivalent state connected adiabatically) belonging to the $(SU(2))_2$ conformal field theory description, which obeys the non-Abelian anyonic statistics appropriate for topological quantum computation [5], provided the 2D samples are not too thin.

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