P H Y S I C A L   R E V I E W   L E T T E R S

Entanglement Spectrum as a Generalization of Entanglement Entropy: Identification of Topological Order in Non-Abelian Fractional Quantum Hall Effect States

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We study the "entanglement spectrum" (a presentation of the Schmidt decomposition analogous to a set of "energy levels") of a many-body state, and compare the Moore-Read model wave function for the $\nu = 5/2$ fractional quantum Hall state with a generic $5/2$ state obtained by finite-size diagonalization of the second-Landau-level-projected Coulomb interactions. Their spectra share a common "gapless" structure, related to conformal field theory. In the model state, these are the *only* levels, while in the "generic" case, they are separated from the rest of the spectrum by a clear "entanglement gap", which appears to remain finite in the thermodynamic limit. We propose that the low-lying entanglement spectrum can be used as a "fingerprint" to identify topological order.

There has been increasing recent interest in using quantum entanglement as a probe to detect topological properties of many-body quantum states [1–4], in particular, the states exhibiting the fractional quantum Hall effect (FQHE) [5–13]. Among various measures of quantum entanglement, the entanglement entropy has by far been the favorite [12–14]. By partitioning a many-body quantum system into two blocks, the entanglement entropy is defined as the von Neumann entropy of the reduced density-matrix of either one of the two blocks, and is a single number that can be obtained from knowledge of the density-matrix eigenvalue spectrum. The density matrix may be written in the form $\hat{\rho} = \exp(-\hat{H})$, so that the entanglement entropy is equivalent to the thermodynamic entropy of a system described by a Hermitian "Hamiltonian" $\hat{H}$ at "temperature" $T = 1$; in the case of a weak entanglement, the "excited states" eigenvalues of $\hat{H}$ are separated from the ground state eigenvalue by a large "energy gap" that becomes infinite in the limit of a simple product state with vanishing entanglement entropy. In this Letter, we point out that the spectrum of the Hamiltonian $\hat{H}$, which we call the "*entanglement spectrum*", reveals much more complete information than the entanglement entropy, a single number.

We examined the entanglement spectrum of the ideal Moore-Read (MR) state with Landau-level filling fraction $\nu = 1/2$, and found that its structure appears to coincide with that of the spectrum of the associated non-Abelian conformal field theory (CFT). In particular, the spectrum is gapless, and (up to a limit determined by the compactification of the system to have a finite area) has the same count of states (characters) at each momentum (Virasoro level), although there is some splitting of the "energies." The MR entanglement spectrum is also "nongeneric", as it contains far fewer levels than the number expected from counting Hilbert-space dimensions; this is a common property of model wave functions, which lack part of the "zero-point fluctuation" of a generic state.

We also examined the "realistic" $\nu = 5/2 = 2 + 1/2$ state obtained by diagonalizing the Coulomb interaction projected into the half-filled second Landau level, and observed that all the extra levels expected in a generic entanglement spectrum are present, but are separated by a finite gap from gapless CFT-like low-lying states with essentially the same structure as those of the Moore-Read state. This gap appears to remain finite in the thermodynamic limit. Thus the low-lying entanglement spectrum is essentially a fingerprint which allows the associated CFT (which characterizes the topological order) to be identified.

If Landau-level mixing is ignored, the $\nu = 5/2$ FQHE system is equivalent to a $\nu = 1/2$ lowest Landau level with interaction pseudopotentials [11] corresponding to a simple Coulomb interaction projected into the second Landau level (we did not refine this to include the quantum well form factor, which may be important for quantitative modeling). In spherical geometry [11], the number of electrons (N_e) and the total number of Landau-level orbitals (N_{orb}) are related by $N_{orb} = 2N_e - 2$. In second-quantization, a many-body state can be written in the basis of orbital occupations; we used the L_z eigenstate basis to divide the spherical surface at a line of latitude into two regions, so the N_{orb} orbitals are partitioned into N_{orb}^A around the north pole, and N_{orb}^B around the south pole (this is spatially the sharpest cut consistent with projection into a Landau level), which is a partition of the Fock space $\mathcal{H}$ into two parts $\mathcal{H}_A \otimes \mathcal{H}_B$.

A Schmidt decomposition (equivalent to the singular value decomposition of a matrix) of a many-body state $|\psi\rangle$ gives

$$|\psi\rangle = \sum_i e^{-(1/2)\xi_i}|\psi_A^i\rangle \otimes |\psi_B^i\rangle, \qquad (1)$$

where $\exp(-\frac{1}{2}\xi_i) \geq 0$, $|\psi_A^i\rangle \in \mathcal{H}_A$, $|\psi_B^i\rangle \in \mathcal{H}_B$, and $\langle \psi_A^i|\psi_A^j\rangle = \langle \psi_B^i|\psi_B^j\rangle = \delta_{ij}$, giving $\exp(-\frac{1}{2}\xi_i)$ as the singular values and $|\psi_A^i\rangle$ and $|\psi_B^i\rangle$ the singular vectors. If the state is normalized, $\sum_i \exp(-\xi_i) = 1$, but it is not necessary to impose the normalization condition.

The ξ_i's are "energy levels" of a system with thermo-dynamic entropy at temperature $T = 1$ equivalent to the entanglement entropy, $\mathcal{S} = \sum_i \xi_i \exp(-\xi_i)$, which has been shown to contain information on the topological properties of the many-body state [12]. The full structure of the entanglement spectrum (logarithmic Schmidt spectrum) of levels ξ_i contains much more information than the entanglement entropy $\mathcal{S}$, a single number. This is analogous to the extra information about a condensed matter system given by its low-energy excitation spectrum rather than just by its ground state energy.

Because the FQHE ground state is translationally and rotationally invariant (with quantum number $L_{\text{tot}} = 0$ on the sphere), and the partitioning of Landau-level orbitals conserves both gauge symmetry and rotational symmetry along the z direction, in either block A or B both the electron number (N_e^A and N_e^B) and the total z-angular momentum (L_z^A and L_z^B) are good quantum numbers constrained by $N_e^A + N_e^B = N_e$, $L_z^A + L_z^B = 0$. The entanglement spectrum splits into distinct sectors labeled by N_e^A and L_z^A.

In the thermodynamic limit, the MR model state can be represented by its "root configuration" [15], which has occupation numbers "11001100···110011", with a repeated sequence ···1100···; in spherical geometry, this is terminated by "11" at both ends. This is the highest-density "MR root configuration", which we define as an occupation-number configuration satisfying a "generalized Pauli principle" that no group of 4 consecutive orbitals contains more than 2 particles (this rule also applies to MR states with quasiholes, and generates the CFT edge spectrum of a finite MR droplet on the open plane [15].) When the MR state is expanded in the occupation-number basis, the only configurations (Slater determinants) present are those obtained by starting from the root configuration, and "squeezing" pairs of particles with $L_z = m_1, m_2$ closer together, reducing $|m_1 - m_2|$, while preserving $m_1 + m_2$ [15].

From the root configuration, we see that there are three distinct ways of partitioning the orbitals: (i) between two 0's; (ii) between two 1's; or (iii) between 0 and 1 (the partitioning between 1 and 0 is equivalent to that between 0 and 1 by reflection symmetry). We use symbols $P[0|0]$, $P[1|1]$, and $P[0|1]$ to represent the three cases, respectively. This will correspond to choosing one of the three sectors of the associated conformal field theory. For finite systems, we always try to draw the boundary of the partitioning either on the equator (if possible), or closest to the equator but in the southern hemisphere. Moreover, we can associate a "natural" value to N_e^A for a particular partitioning, i.e., the total number of 1's in the root occupation sequence on the left-hand side to the boundary. In this Letter, it is sufficient to consider only levels whose N_e^A is exactly this natural value. Table I describes the precise meaning of these symbols for systems that are considered here.

TABLE I. The numbers in the parenthesis are values of (N_{orb}^A, N_e^A), respectively, for each system and partitioning as specified.

N_e	$P[0|0]$	$P[0|1]$	$P[1|1]$
10	(7,4)	(8,4)	(9,5)
12 or 14	(11,6)	(12,6)	(13,7)
16	(15,8)	(16,8)	(17,9)

Figure 1 shows the spectra for each of the three different ways of partitioning, for the Moore-Read state at $N_e = 16$ and $N_{\text{orb}} = 30$. The spectrum not only has far fewer levels than expected for a generic wave function, but also exhibits an intriguing level-counting structure (as a function of L_z^A and N_e^A) that resembles that of the associated conformal field theory of the edge excitations. Intuitively, this is because the boundary of the Landau-level partitioning indeed defines an edge shared by region A and B.

In the intuitive picture, the quantum entanglement between A and B arises from correlated quasihole excitations across the boundary along which the partitioning is carried
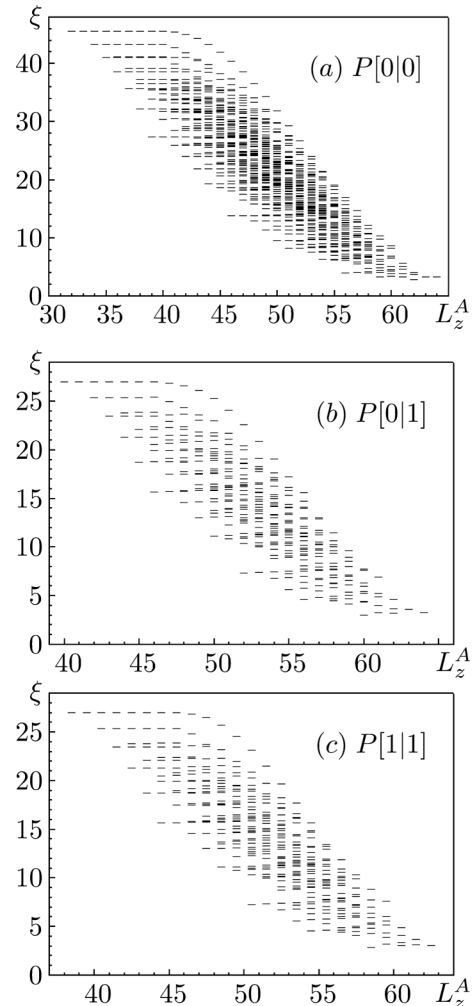

FIG. 1. The complete entanglement spectra of the $N_e = 16$ and $N_{\text{orb}} = 30$ Moore-Read state (only the relative values of ξ and L_z^A are meaningful).

out. Any quasihole excitation in region A necessarily pushes electrons into region B, and vice versa. However, the electron density anywhere on the sphere must remain constant, which can be achieved if the quasihole excitations in A and B are correlated (entangled). This gives the empirical rules of counting the levels. Take the spectrum in Fig. 1(a) as an example. The partitioning $P[0|0]$ results in the root configuration 110011001100110 on the northern hemisphere (region A), and it corresponds to the single "level" at the highest possible value of $L_z^A = L_{z,\max}^A = 64$. We measure the L_z^A by its deviation from $L_{z,\max}^A$, i.e. $\Delta L := L_{z,\max}^A - L_z^A$, which has the physical meaning of being the total z-angular momentum carried by the quasiholes. At $\Delta L = 1$, the levels correspond to edge excitations upon the $\Delta L = 0$ root configuration. There is exactly one edge mode in this case, represented by the MR root configuration 110011001100101.

The number of $\Delta L = 2$ levels can be counted in exactly the same way. There are three of them, of which the root configurations are

$$1100110011001001$$
$$1100110011000110 \, ,$$
$$1100110010101010$$

while for $\Delta L = 3$, the five root configurations are

$$11001100110010001$$
$$11001100110001010$$
$$11001100101001100 \, .$$
$$11001100101010010$$
$$11001010101010100$$

The counting for the levels at small ΔL for $P[0|1]$ and $P[1|1]$ can be obtained similarly.

For an infinite system in the thermodynamic limit, the above idea gives an empirical counting rule of the number of levels at any ΔL; i.e., it is the number of independent quasihole excitations upon the semi-infinite root configuration uniquely defined by the partitioning. For a finite system, this rule explains the counting only for small ΔL; for large ΔL, the finite-size limits the maximal angular momentum that can be carried by an individual quasihole. Therefore the number of levels at large ΔL in a finite system will be smaller than the number expected in an infinite system. Not only is this empirical rule consistent with all our numerical calculation, but it also explains why $P[0|0]$ and $P[0|1]$ have essentially identical low-lying structures. This is because the (semi-infinite) configuration "$\cdots 1100110$" is essentially equivalent to "$\cdots 11001100$" (with an extra "0" attached to the right). We expect that $P[0|0]$ and $P[0|1]$ become exactly identical in the thermodynamic limit.

For completeness, we list the root configurations associated with the first few low-lying levels in Fig. 1(c).

$$\Delta L = 0 : \quad 11001100110011001$$
$$\Delta L = 1 : \quad 110011001100110001$$
$$110011001100101010$$
$$\Delta L = 2 : \quad 1100110011001100001$$
$$1100110011001010010$$
$$1100110011001001100$$
$$1100110010101010100$$

Figure 2 shows the spectra of the system of the same size as in Fig. 1, i.e., $N_e = 16$ and $N_{\mathrm{orb}} = 30$, but for the ground state of the Coulomb interaction projected into the second Landau level, obtained by direct diagonalization. Interestingly, the low-lying levels have the same counting structure as the corresponding Moore-Read case. We identify these low-lying levels as the "CFT" part of the spec-
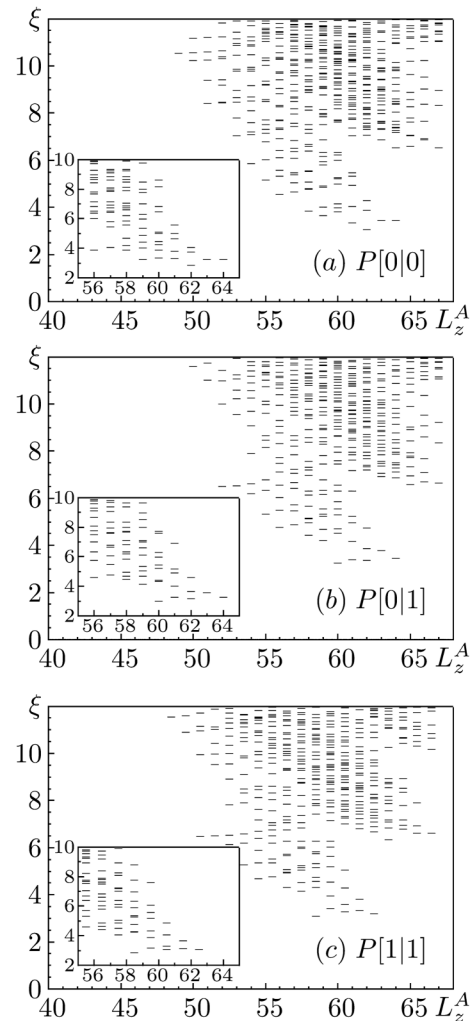

FIG. 2. The low-lying entanglement spectra of the $N_e = 16$ and $N_{\mathrm{orb}} = 30$ ground state of the Coulomb interaction projected into the second Landau level (there are levels beyond the regions shown here, but they are not of interest to us). The insets show the low-lying parts of the spectra of the Moore-Read state, for comparison [see Fig. 1]. Note that the structure of the low-lying spectrum is essentially identical to that of the ideal Moore-Read state.
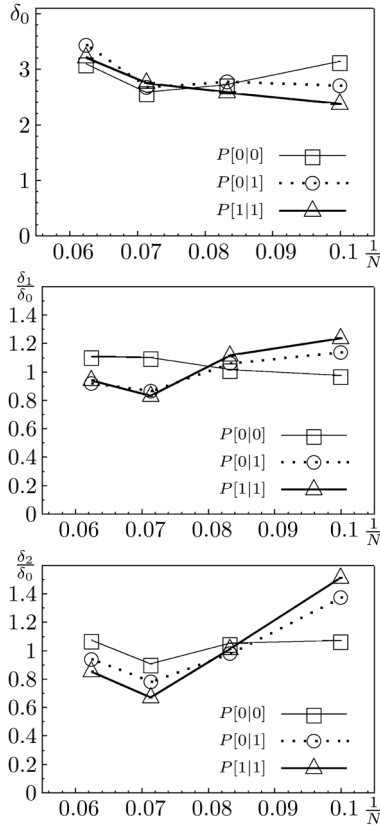
FIG. 3. Entanglement gap as a function of $1/N$. δ_0 is the gap at $\Delta L = 0$, i.e., the distance from the single CFT level at $\Delta L = 0$ to the bottom of the generic (non-CFT) levels at $\Delta L = 0$. At $\Delta L = 1$, 2, the gap $\delta_{1,2}$ is defined as the distance from the average of the CFT levels to the bottom of the generic levels. See Table I for the details of various partitionings.

trum, in contrast to the other *generic*, non-CFT levels that are expected for generic many-body states. At relatively small ΔL (up to a limit which grows with the size of the system), the CFT levels are separated from the generic levels by a clear gap, which we define as the distance from the average of the CFT levels to the bottom of the generic levels.

Figure 3 shows the value of this gap (at $\Delta L = 0$, 1, 2, respectively) as a function of the size of the system, based on which we speculate that the gap between the CFT and non-CFT levels remains finite in the thermodynamic limit for all ΔL. The observed fact that the structure of the low-lying spectrum is essentially identical to that of the Moore-Read state, as well as the existence of the "entanglement gap", serve as evidence that the realistic $\nu = 5/2$ FQH states is indeed modeled by the Moore-Read state.

Assuming that the gap does remain finite in the thermo-dynamic limit, characterization of the entanglement spectrum is a reliable way to identify a topologically ordered state. While finite-size numerical studies often show impressive (e.g., 99%) overlaps between model wave functions (Laughlin, Moore-Read, etc.) and realistic states at intermediate system sizes, this cannot persist in the thermodynamic limit. Furthermore, the entanglement spec-

trum is a property of the ground state wave-function itself, as opposed to the physical excitations of a system with boundaries, so allows direct comparison between model states and physical ones.

The asymptotic behavior of the characters of a CFT (the *count* of independent states at each Virasoro level, in each sector) defines both the effective conformal anomaly $\tilde{c}$ of the CFT, and the quantum dimension of each sector. To the extent that there is a clear separation of the gapless, low-lying CFT-like modes and the generic (but "gapped") modes, one can count the number of gapless modes as a function of momentum parallel to the boundary separating the two regions. For a finite-size system, these will match the CFT characters up to some limit that grows with system size. These numbers are integers, so are not subject to numerical error, and in principle, both $\tilde{c}$ and the quantum dimensions can be extracted from their behavior as the system size grows.

As a critical point is approached, the "entanglement gap" may still be finite but well below the temperature $T = 1$ at which the von Neumann entropy is evaluated. We suggest that the direct study of the low-lying entanglement spectrum is a far more meaningful way to characterize bipartite entanglement. Equivalently, the $T \to 0$ "low-temperature" entropy of the modified family of density matrices $\hat{\rho}^{(1/T)}$ may prove useful, as this corresponds to the thermodynamic entropy of the entanglement spectrum at temperature T.

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