Model Fractional Quantum Hall States and Jack Polynomials

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We describe an occupation-number-like picture of fractional quantum Hall states in terms of polynomial wave functions characterized by a dominant occupation-number configuration. The bosonic variants of single-component Abelian and non-Abelian fractional quantum Hall states are modeled by Jack symmetric polynomials (Jacks), characterized by dominant occupation-number configurations satisfying a generalized Pauli principle. In a series of well-known quantum Hall states, including the Laughlin, Read-Moore, and Read-Rezayi, the Jack polynomials naturally implement a "squeezing rule" that constrains allowed configurations to be restricted to those obtained by squeezing the dominant configuration. The Jacks presented in this Letter describe new trial uniform states, but it is yet to be determined to which actual experimental fractional quantum Hall effect states they apply.

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The Laughlin wave function [1] has provided the key to understanding the physics of the fractional quantum Hall (FQH) effect: It accurately models the simplest Abelian FQH states and is the building block of model wave functions for more general states, both Abelian and (using cluster projections) non-Abelian ones, such as the Moore-Read [2] and Read-Rezayi [3] states. Apart from trivial (Gaussian) factors which we will drop, such model wave functions are conformally invariant multivariable polynomials $\psi(z_1, \ldots, z_N)$; despite their explicit availability, analytic calculations of correlation functions and other physical properties have not so far been possible because of the intractability of their expansions in the noninteracting basis of occupation-number states (Slater determinants or monomials). The simplest physically relevant model FQH states are antisymmetric polynomials, describing spin-polarized electrons in a partially filled Landau level with no internal "pseudospin" degrees of freedom, but it is also useful to study symmetric (bosonic) FQH wave functions from which they are obtained by multiplication by odd powers of the Vandermonde determinant.

In this Letter, we describe a unified occupation-basis framework for the description of many model one-component FQH states in terms of the Jack symmetric polynomial(s) ["Jack(s)"] [4]. The Jacks naturally implement a type of "generalized Pauli principle" on a generalization of Fock spaces for Abelian and non-Abelian fractional statistics [5]. We note that (bosonic) Laughlin, Moore-Read, and Read-Rezayi wave functions (as well as others, such as the state that Simon *et al.* [6] have called the "Gaffnian") can be explicitly written as Jack symmetric polynomials, which have known (recursively defined) expansions in monomials (free-boson occupation-number states), and have rich algebraic properties. These uniform FQH condensate wave functions can be obtained by requiring that a Jack simultaneously obeys highest-weight (HW, absence of quasiholes) and lowest-weight (LW, absence of quasiparticles) conditions. The Jacks described in the present Letter also provide new FQH wave functions at arbitrary fillings $\nu = \frac{k}{r}$, with k and r integers. The generalized Pauli principle and the wave functions introduced here allow for: counting of the dimension of n-quasihole Hilbert space; degeneracy on the torus; specific heat calculations; and electron and quasihole propagators on the edge of the liquid. It is now known [7] that some of the Jacks (e.g., at filling $\nu = 2/5$ or $3/7$) have very good overlap >0.95 with the Coulomb ground state and the composite Fermion wave function [8] for up to 15 particles.

Jacks $J_\lambda^\alpha(z)$ are symmetric polynomials in $z \equiv \{z_1, z_2, \ldots, z_N\}$, labeled by a partition λ with length $\ell_\lambda \leq N$, and a parameter α; λ can be represented as a (bosonic) occupation-number configuration $n(\lambda) = \{n_m(\lambda), m = 0, 1, 2, \ldots\}$ of each of the lowest Landau level orbitals $(2\pi m! 2^m)^{-1/2} z^m \exp(-|z|^2/4)$ with angular momentum $L_z = m\hbar$ (see Fig. 1), where, for $m > 0$, $n_m(\lambda)$ is the multiplicity of m in λ. When $\alpha \to \infty$, $J_\lambda^\alpha \to m_\lambda$, which is the monomial wave function of the free-boson state with

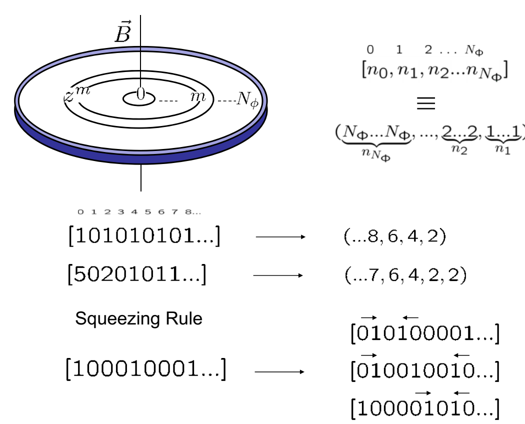

FIG. 1 (color online). Upper half: The Landau problem on a disk. The occupation basis provides for the number of particles n_m in the orbital of angular momentum m. Lower half: Examples of occupation to monomial basis conversion and squeezing rule.

occupation-number configuration $n(\lambda)$; a key property of the Jack J_λ^α is that its expansion in terms of monomials contains only terms m_μ, where $\mu < \lambda$ means the partition μ is *dominated* by λ [4]. Jacks are also eigenstates of a Laplace-Beltrami operator $\mathcal{H}_{\rm LB}(\alpha)$ given by

$$\sum_i \left(z_i \frac{\partial}{\partial z_i}\right)^2 + \frac{1}{\alpha}\sum_{i<j}\frac{z_i + z_j}{z_i - z_j}\left(z_i\frac{\partial}{\partial z_i} - z_j\frac{\partial}{\partial z_j}\right). \quad (1)$$

We note that the bosonic Laughlin state $\psi_L^{(r)}$ at filling $\nu = 1/r$, r even, is a Jack polynomial:

$$\psi_L^{(r)} = \prod_{i<j}^N (z_i - z_j)^r = J^{\alpha_{1,r}}_{\lambda^0(1,r)}(z), \qquad \alpha_{k,r} \equiv -\frac{k+1}{r-1}, \quad (2)$$

which is the $k = 1$ case of a Jack defined for any positive integer k so that $N = k\bar{N}$, and $n_m(\lambda^0(k,r)) = k$ for $m = (j-1)r$, $j = 1, 2, \ldots, \bar{N}$, with $n_m = 0$ otherwise. Note that $\lambda^0(k,r)$ is the "(k,r,N)-admissible" partition [9] that minimizes $|\lambda| \equiv M \equiv \sum_m m n_m$ at fixed N [λ is (k,r,N)-admissible if $n(\lambda)$ obeys a generalized Pauli principle where, for all $m \geq 0$, $\sum_{j=1}^r n_{m+j-1} \leq k$, so r consecutive "orbitals" contain no more than k particles]. Note that here the Jack parameter $\alpha_{k,r}$ is a *negative rational*; study of symmetric Jacks of this type was recently initiated in Ref. [9]. Earlier work generally assumes that α is a positive real (Jacks with real $\alpha > 0$ and unrestricted λ occur in the solution of the integrable Calogero-Sutherland model [10]). Nonsymmetric Jack polynomials can also describe spin-fractional quantum Hall effect wave functions such as Halperin and Haldane-Rezayi [11].

It is straightforward to see that $\psi_L^{(r)}$ is a Jack: It has the obvious property that it is annihilated by operators

$$D_i^{L,r} = \frac{\partial}{\partial z_i} - r\sum_{j(\neq i)}{}' \frac{1}{z_i - z_j}; \qquad D_i^{L,r}\psi_L^{(r)} = 0. \quad (3)$$

It is then also annihilated by the combination $\sum_i z_i D_i^{L,-1} z_i D_i^{L,r}$, which is equal to $\mathcal{H}_{\rm LB}(\alpha_{1,r})$ minus a constant (found by direct calculation to be $\frac{1}{12} rN(N-1) \times [N + 1 + 3r(N-1)]$), so $\psi^{(r)}$ is an eigenstate of $\mathcal{H}_{\rm LB}(\alpha_{1,r})$. It is now easy to identify the dominant configuration $n(\lambda^0_{1,r})$ and verify that the eigenstate is nondegenerate, confirming Eq. (2). For $r = 2$, this also follows implicitly from Ref. [9], where it was shown that the set of Jacks with parameter $\alpha_{k,2}$ and $(k,2,N)$-admissible λ is a basis for the space of symmetric polynomials that vanish when $k+1$ variables z_i coincide. The space of symmetric polynomials space can be divided into subspaces of fixed $M = |\lambda|$, and, for $|\lambda| = \lambda^0_{k,r}$, there is a *single* (k,r,N)-admissible partition, so a polynomial with the appropriate properties is unique and must be a Jack.

It is useful to identify the "dominance rule" (a partial ordering of partitions $\lambda > \mu$) with the "squeezing rule" [10] that connects configurations $n(\lambda) \to n(\mu)$: Squeezing is a two-particle operation that moves a particle from orbital m_1 to m_1' and another from m_2 to m_2', where $m_1 < m_1' \leq m_2' < m_2$ and $m_1 + m_2 = m_1' + m_2'$; $\lambda > \mu$ if $n(\mu)$ can be derived from $n(\lambda)$ by a sequence of squeezings (see Fig. 1). This means that when model FQH wave functions equivalent to Jacks are expanded in basis of occupation-number states, only configurations obtained by squeezing from a dominant configuration will be present (this crucial property persists in fermionic model FQH wave functions given by the product of a Jack with a power of the Vandermonde determinant).

Jacks can be normalized so that

$$J_\lambda^\alpha = m_\lambda + \sum_{\mu<\lambda} v_{\lambda\mu}(\alpha) m_\mu. \quad (4)$$

The coefficients $v_{\lambda\mu}(\alpha)$ are (recursively) known [12]; they are finite and real positive for real $\alpha > 0$ and are holomorphic functions of α except for poles at a (λ, μ)-dependent set of negative rational values [9]. Feigin *et. al.* [9] proved that, for the (k,r,N)-admissible partitions, $v_{\lambda\mu}(\alpha)$ is analytic at $\alpha_{k,r}$, and the set of admissible Jacks with this parameter forms a basis of a differential ideal $I_N^{k,r}$ in the space of symmetric polynomials. This requires that $(k+1)$ and $(r-1)$ (but *not necessarily* k and r) be coprime. For the case $r = 2$, and k integer, these polynomials are a basis for the $\nu = k/r = k/2$ bosonic non-Abelian Read-Rezayi FQH states with quasiholes, with special cases $k = 1$ (Laughlin state) and $k = 2$ (Moore-Read state). By multiplying these wave functions by $\psi_L^{(m)}$, this generalizes to the $\nu = k/(km + 2)$ Read-Rezayi states and reproduces the generalized Pauli principle exclusion statistics structure found empirically in numerical studies by one of us [5].

The $\nu = 1/r$ Laughlin state is a Jack polynomial with parameter $\alpha_{1,r}$ and $n(\lambda) = [10^{r-1}10^{r-1}\ldots]$, where "$0^{r-1}$" means a sequence of $r-1$ "empty orbitals." A basis of one-quasihole states can similarly be shown to be given by the Jack with $n(\lambda) = [10^{r-1}1\ldots 0^{r-1}10^r10^{r-1}\ldots]$, where there is a single extra empty orbital. These states all have different M, and hence are orthogonal, and form a multiplet. It is easily seen that there are $N+1$ such occupation numbers. The Laughlin quasihole at position z_{qh} is obtained as a coherent state superposition of the previous Jacks with coefficients $(-z_{qh})^i$, with $i = 0, \ldots, N$. A linearly independent basis of two-quasihole state is given by Jacks with the same α and two extra empty orbitals in $n(\lambda)$. For example, at $r = 2$, two such configurations $n(\lambda)$ and $n(\lambda')$ (with the same M) are $[10100010101\ldots]$ and $[10010100101\ldots]$. While m_λ and $m_{\lambda'}$ are orthogonal free-boson wave functions, the Jack FQH wave functions $J_\lambda^{\alpha_{1,2}}$ and $J_{\lambda'}^{\alpha_{1,2}}$ are not orthogonal with respect to the usual quantum-mechanical scalar product. This highlights an important difference between the basis of "admissible Jacks" (with a generalized Pauli principle) and the ordinary free-particle basis: The pure Jack wave functions are not eigenstates of a Hermitian Hamiltonian [$\mathcal{H}_{\rm LB}(\alpha)$ is not Hermitian for finite α] and are linearly independent but not orthogonal. [In contrast, Jacks with α real positive (and unrestricted λ) are orthogonal with respect to a combinatorically motivated scalar product [4] and also as Calogero-Sutherland model wave functions.]

Partitions λ can be classified by λ_1, their largest part. When J_λ^α is expanded in occupation-number states (monomials), no orbital with $m > \lambda_1$ is occupied, and Jacks with $\lambda_1 \leq N_\Phi$ form a basis of FQH states on a sphere surrounding a monopole with charge N_Φ [13]. Uniform states on the sphere satisfy conditions $L^+\psi = 0$ (HW) and $L^-\psi = 0$ (LW), where $L^+ = E_0$ and $L^- = N_\Phi Z - E_2$, where $Z \equiv \sum_i z_i$, and $E_n = \sum_i z_i^n \partial/\partial z_i$. When both conditions are satisfied, $E_1\psi \equiv M\psi = \frac{1}{2}NN_\Phi\psi$. It is very instructive to find the conditions for a Jack to satisfy the HW condition $E_0 J_\lambda^\alpha = 0$. The action of E_0 on a Jack can be obtained from a formula due to Lassalle [14]: By using the property that, for real $\alpha > 0$, all of the $v_\lambda(\alpha)$ are real positive [4], the HW condition can be satisfied only for real $\alpha < 0$. Another condition we find is that $n_0 \equiv N - \ell_\lambda > 0$ (nonzero occupancy of the $m = 0$ orbital). We then find that a necessary (but not sufficient) condition is

$$N - \ell_\lambda + 1 + \alpha(\lambda_\ell - 1) = 0, \tag{5}$$

where λ_ℓ is the smallest (nonzero) part in λ. This imposes the following two conditions: (i) α is a negative rational, which we can choose to write as $-(k+1)/(r-1)$, with $(k+1)$ and $(r-1)$ both positive, and relatively prime; (ii) $\lambda_\ell = (r-1)s + 1$, and $n_0 = (k+1)s - 1$, where $s > 0$ is a positive integer. The remaining HW conditions require that all parts in λ have multiplicity k, so that $n(\lambda) = [n_0 0^{s(r-1)} k 0^{r-1} k 0^{r-1} k \ldots]$ [i.e., the (k, r, N)-admissibility condition is satisfied as an equality for orbitals $m \geq \lambda_\ell$]. The case $s = 1$ gives the FQH ground states which also obey the LW condition, with filling $\nu = k/r$, while the cases $s > 1$ are intimately related to what we interpret as the *quasiparticle* (not quasihole) excitations of these $\nu = k/r$ states, where $r = 2$ corresponds to the bosonic Laughlin–Moore-Read–Read-Rezayi sequence (see Fig. 2). The $s > 1$ case represents a new mathematical result, investigated elsewhere [15].

We will describe the quasiparticle construction elsewhere but note that, when it is applied to the case $k = 1$, $r = 2$ ($\nu = 1/2$ bosonic Laughlin state), it reproduces the model quasiparticle state given by Jain's projective construction [8]. The above derivation very simply reproduces

Solutions of $L^+ J_\lambda^\alpha = 0$

	$k=1,\ r=2$	$k=2,\ r=2$	$k=3,\ r=2$
$s=1$	$\vert 101010101\ldots\rangle$	$\vert 202020202\ldots\rangle$	$\vert 303030303\ldots\rangle$
$s=2$	$\vert 300101010\ldots\rangle$	$\vert 500202020\ldots\rangle$	$\vert 700303030\ldots\rangle$
	⋮	⋮	⋮
	$k=1,\ r=3$	$k=2,\ r=3$	$k=3,\ r=4$
$s=1$	$\vert 10010010010\ldots\rangle$	$\vert 2002002002\ldots\rangle$	$\vert 3000300030003\ldots\rangle$
$s=2$	$\vert 30000100100\ldots\rangle$	$\vert 5000020020\ldots\rangle$	$\vert 7000000300030\ldots\rangle$
	⋮	⋮	⋮

FIG. 2. Solutions to $L^+ J_\lambda^\alpha =$ are parametrized by one integer, $s > 0$. $s = 1$ states are FQH ground states. $s > 1$ states are related to quasiparticles.

the admissibility conditions found in Ref. [9] and shows that the (k, r, N)-admissible Jacks that minimize M at fixed $N = k\bar{N}$ are the only pure Jacks that are acceptable FQH wave functions. As the $L^{+,-}$ are single-body operators, the HW and LW conditions will generalize to the fermionic states obtained by multiplication with powers of Vandermonde determinants.

We now turn our attention to the Moore-Read state [2,16]. It was introduced as a model for the observed $\nu = 5/2$ spin-polarized FQH ($\nu = 1/2$ in the second Landau level) and is the $m = 1$ case of the $\nu = 2/(2m+2)$ state

$$\Psi_{\rm MR}^m = \prod_{i<j}(z_i - z_j)^{m+1}\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right), \tag{6}$$

where $\Psi_{\rm MR}^0$ is a $\nu = 1$ FQH state of bosons at $\nu = k/r$, where $k = r = 2$: The exclusion statistics picture of this bosonic state is $n(\lambda^0(2,2)) = [20202\ldots]$ [5] or (the highest density state with) not more than 2 particles in 2 consecutive orbitals. It was initially defined as the correlation function of an Ising Majorana field $\psi_{(2,1)} = \psi(z)$ with scaling dimension $h_{2,1} = \frac{1}{2}$ in the minimal model $M(4,3)$ with $c = \frac{1}{2}$ [small indices label degenerate fields in conformal field theory (CFT)]. The correlation functions of a field $\psi_{(m,n)}$ satisfy an nmth-order differential equation. This allows us to define a set of N annihilation operators. For the Pfaffian $\langle\psi(z_1)\psi(z_2)\ldots\psi(z_N)\rangle = \mathrm{Pf}(\frac{1}{z_i - z_j})$, the annihilation operators are [17]

$$D_i^{\rm Pf} = \frac{\partial^2}{\partial z_i^2} - \sum_{j\neq i}\frac{A_{2,1}}{z_i - z_j}\frac{\partial}{\partial z_j} - \sum_{j\neq i}\frac{B_{2,1}}{(z_i - z_j)^2}, \tag{7}$$

where $A_{2,1} = 2(2h_{2,1} + 1)/3$ and $B_{2,1} = h_{2,1}A_{2,1}$. The Pfaffian satisfies $D_i^{\rm Pf}\mathrm{Pf}(\frac{1}{z_i - z_j}) = 0$. According to the general prescription for obtaining FQH wave functions out of CFT correlators, the first bosonic Moore-Read state $\psi_{\rm MR}^0$ is obtained by multiplying the Pfaffian by a Vandermonde factor: $\psi_L^{(1)}$. It is straightforward to transform $D_i^{\rm Pf}$ to obtain operators $D_i^{\rm MR}$ that annihilate $\psi_{\rm MR}$ and show that $\sum_i D_i^{\rm MR}$ is $\mathcal{H}_{\rm LB}(-3)$ plus a constant [found by direct computation to be $-N(16 - 18N + 5N^2)/18$], which confirms that $\psi_{\rm MR}^0 = J_{\lambda^0(2,2)}^{-3}$.

The $\nu = k/2$ bosonic Read-Rezayi (RR) states [3] are "Z_k parafermion states." The first RR state is related to the Z_3 Potts model [18] and is annihilated by a third-order differential operator. The dominant configuration of this state is $n(\lambda^0(3,2)) = [3030303\ldots]$ or (the highest density state with) not more than 3 particles in 2 consecutive orbitals. The Z_3 parafermion FQH state is a single Jack and diagonalizes the second-order Laplace-Beltrami operator. The RR Z_k sequence is $\psi_{RR}^0(z) = J_{\lambda^0(k,2)}^{\alpha(k,2)}(z)$.

A bosonic state at $\nu = 2/3$ (or a fermionic one with $\nu = 2/5$) has been referred to as a Gaffnian [6]. The dominant configuration of the bosonic state ψ_G^0 is $n(\lambda^0(2,3)) = [2002002002\ldots]$ or the highest density $(2,3)$ state. We find that this state is annihilated by (7) with $h_{2,1} = 3/4$, as expected, as the wave function of this state is also the correlation function of a minimal CFT $M(5, 3)$ field $\psi_{(2,1)}$ with this scaling dimension [6], and we identify $\psi_G^0(z)$ as the (2, 3) vacuum Jack $J^{\alpha_{2,3}}_{\lambda^0(2,3)}(z)$.

Instead of using the differential equations that they satisfy, it is easier to identify the FQH states with Jacks from their clustering properties. Information on how the Jacks vanish as $k+1$ coordinates coincide is needed: We verified that, for *any* (k, r, N)-admissible λ, if $z_1 = z_2 = \cdots = z_k = Z$, $J^{\alpha_{k,r}}_{\lambda}(z)$ has a factor $\prod_{i=k+1}^{N}(Z - z_i)^r$, showing how it vanishes as a cluster of $k+1$ coincident coordinates is formed. This agrees with the properties of the bosonic Laughlin–Read-Rezayi states ($k \geq 1$, $r = 2$), as well as the Gaffnian ($k = 2$, $r = 3$). For the case of the FQH ground states, where $\lambda = \lambda^0_{k,r}$, the Jacks satisfy a stronger clustering property that relates N- and $(N+k)$-particle states: For $\{z\} \equiv \{z_1, \ldots, z_N\}$,

$$\prod_{i=1}^{N}(Z - z_i)^r J^{\alpha_{k,r}}_{\lambda^0(k,r)}(\{z\}) = J^{\alpha_{k,r}}_{\lambda^0(k,r)}(\{z\}, Z, \ldots, Z), \tag{8}$$

where on the right-hand side, $z_i = Z$ for $i = N+1, \ldots, N+k$. As a corollary, when a (k, r) Jack FQH ground state is fully k-clustered, i.e., $z_{ki-j} \to Z_i$ for $i = 1, \ldots, \bar{N}$ and $j = 1, \ldots, k$, it becomes a Laughlin state in the cluster coordinates $J^{\alpha_{k,r}}_{\lambda^0(k,r)}(z) \to \psi_L^{(kr)}(Z_i)$. For these model FQH states, removing a cluster of k particles at a point Z is exactly equivalent to inserting r flux quanta (or vortices) at that point (as is well known in the $k = 1$ Laughlin case and implicit in the Read-Rezayi construction of the parafermion states [3]).

As is obvious from their clustering property, the (k, r, N)-admissible Jacks also have the property that a $(k+1)$ cluster of particles cannot have relative angular momentum less than r and hence are simultaneous null states of Hermitian operators $\hat{H}^{(k+1)}_{r-1}$, which are the $(k+1)$-body generalizations [6] of two-body Hamiltonians $\hat{H}^{(2)}_{r-1}$ where the only nonzero two-body pseudopotentials [13] are $V_m > 0$ for $m \leq r-1$. However, for $k > 1$, $r > 3$ [and $(k+1)$ and $(r-1)$ relatively prime], the number of linearly independent null states of $\hat{H}^{(k+1)}_{r-1}$ is larger than the set of (k, r, N)-admissible Jacks, and, in particular, the homogeneous $\nu = k/r$ FQH state with $\lambda = \lambda^0(k, r)$ is not in general unique, as seen in Table I of Ref. [19]. For example, for $(k, r) = (3, 4)$ [spinless boson states with $N = 3\bar{N}$, $N_\Phi = 4(\bar{N} - 1)$], the $(3, 4, N)$-admissible Jack with $\lambda = \lambda^0(3, 4)$ is a zero-mode eigenstate of $\hat{H}^{(4)}_3$ but is not unique. We did not find any other local Hermitian n-body pseudopotential operators that could be added to $\hat{H}^{(4)}_3$ to make this Jack a unique null state, so it remains unclear how to define the $k > 1$, $r > 3$ Jack FQH states as unique null states of a model Hamiltonian (although requiring them to also be eigenstates of the non-Hermitian operator $\mathcal{H}_{\rm LB}$ does make them unique). Fermionic states at filling $k/(km + r)$ will also be zero modes of two-body operators with angular momentum m. Mathematically, they are related to correlation functions of primary fields of nonunitary CFTs [6,9].

In conclusion, we have identified a number of model bosonic FQH ground states at $\nu = k/r$ (with $k+1$ and $r-1$ relatively prime) with a set of special Jack symmetric polynomials, whose expansion in monomials (free-particle occupation-number states) is (recursively) known. The FQH states described here, being single Jacks, are eigenstates of a multiplet of N mutually commuting, higher derivative many-body operators (Sekiguchi operators) [20,21], of which the first is $E_1 = M$, the total momentum, and the second is $\mathcal{H}_{\rm LB}(\alpha)$. We obtained the (k, r, N)-admissible partitions [9] as a special case of the highest-weight conditions on the Jacks that generalizes to a (k, r, s, N) admissibility to include quasiparticle states [15]. By using the formalism and the results described here, we will be able to address several issues: the description of non-Abelian FQH quasiparticles (not quasiholes), the computation of the specific heat of the (k, r) sequence of quantum Hall states, the comparison of the (k, r) states and the Jain sequence of FQH states, as well as a computation of electron and quasihole propagators of the Jack FQH edge states.

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