

Mass Effects in Muon and Semileptonic $b \rightarrow c$ Decays

Alexey Pak and Andrzej Czarnecki

Department of Physics, University of Alberta Edmonton, AB T6G 2G7, Canada

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Quantum chromodynamics (QCD) effects in the semileptonic decay $b \rightarrow c\ell\bar{\nu}$ are evaluated to the second order in the coupling constant, $\mathcal{O}(\alpha_s^2)$, and to several orders in the expansion in quark masses, m_c/m_b . Corrections are calculated for the total decay rate as well as for the first two moments of the lepton energy and the hadron system energy distributions. Translated into QED and applied to the muon decay, they decrease its predicted rate by -0.43 ppm.

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Decays of heavy fermions are an abundant source of information about fundamental interactions. Particularly important among them is the muon (μ) decay. Insensitive to strong interactions, it can be very precisely described by the electroweak model. The experiment MuLan at the Paul Scherrer Institute will likely measure the rate of the muon decay with an uncertainty better than 1 ppm and thus improve the determination of the Fermi constant G_F that describes the strength of the charged-current weak interaction [1]. Along with the fine structure constant α and the Z-boson mass, G_F is one of the three pillars of electroweak Standard Model tests [2].

In a separate effort, the TWIST experiment at TRIUMF measures energy and angular distributions of positrons in the μ^+ decay, testing the standard model and searching for new interactions, notably new bosons predicted by left-right symmetric models [3–5].

To match this experimental progress, both the rate [6] and the energy distribution [7] have been calculated in quantum electrodynamics (QED) with $\mathcal{O}(\alpha^2)$ accuracy. Two-loop weak corrections have also been calculated [8]. In the decay rate studies, the electron mass m_e was assumed negligible in the already small $\mathcal{O}(\alpha^2)$ effects.

Here we show that the finite m_e effect decreases the muon decay rate by about half ppm, exceeding previous estimates [9] and approaching the expected MuLan precision.

The final-state fermion mass effects are much larger in the heavy-quark decay $b \rightarrow c\ell\bar{\nu}$. Studied in B factories and the Tevatron, this process provides information about the Cabibbo-Kobayashi-Maskawa (CKM) matrix element V_{cb} , as well as about parameters governing heavy-quark dynamics (see [10] for an up to date review and references). Also in this case, theoretical studies at $\mathcal{O}(\alpha_s^2)$ are complete only for a massless final-state quark [11]. For the actual massive c -quark, $\mathcal{O}(\alpha_s^2)$ effects are known in some special cases of kinematics [12,13]. So-called Brodsky-Lepage-Mackenzie (BLM) corrections [14] have been obtained for the width [15], moments of the energy spectrum [16,17], and triple differential distributions [18]. Also some logarithms of the mass m_c have been determined to all orders in α_s [19]. Most recently, Melnikov calculated numerically

the m_c effects for the width and the first two moments of the energy distribution of hadrons and of the charged lepton produced in this decay [20]. In this paper we present corresponding analytical results obtained as an expansion in powers and logarithms of $\rho \equiv m_c/m_b$ [21,22].

The construction of such mass expansion is illustrated with an $\mathcal{O}(\alpha_s)$ example in Fig. 1. Real and virtual corrections are calculated together as cuts of the diagram 1(a). Depending on virtualities of momenta flowing into and through the charm-quark lines, m_c can be treated as small compared to those momenta, or else those momenta can be treated as small compared to the b -quark mass in other lines of the diagram. The most interesting case is shown in Fig. 1(d), where two-loop momenta are of order m_c . This configuration generates odd powers of ρ and will be discussed in some detail below.

At $\mathcal{O}(\alpha_s^2)$ the number of diagrams is larger and the analysis of momentum scales more challenging, with as many as 11 regions in some diagrams, but it follows the general pattern outlined above. As a result, even the four-loop diagrams shown in Fig. 2 are calculated as a series in ρ with exact coefficients. The most challenging contributions arise when all loop momenta are hard ($\sim m_b$). All charm-quark lines are Taylor-expanded in m_c , generating a large number of integrals with various powers of denominator factors. The method of Ref. [23] is employed to reduce all integrals to a set of master integrals. Most are

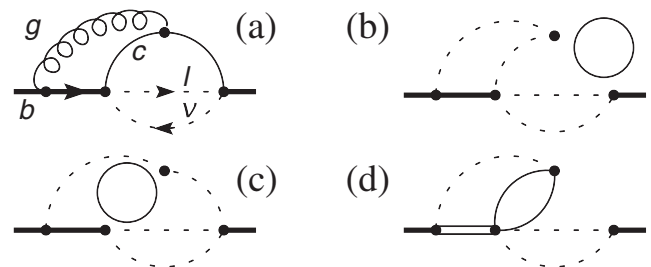


FIG. 1. Example of an expansion of a double-scale integral. Thick, thin, and dashed lines correspond to m_b , m_c , and massless propagators. Loop momenta can be all hard [Taylor expansion in m_c of (a)], and one or more soft [(b),(c), and (d)]. Double line in (d) denotes a static propagator.

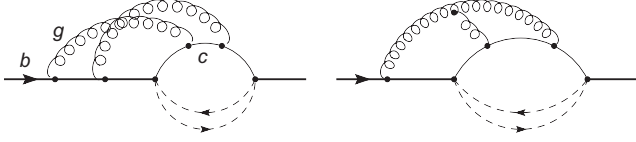


FIG. 2. Examples of diagrams contributing to the $\mathcal{O}(\alpha_s^2)$ corrections to the b -quark decay rate.

known [9], sufficing to reproduce the $m_c = 0$ limit as well as the $\mathcal{O}(\rho^2)$ terms. Unfortunately, the following term $\mathcal{O}(\rho^4)$ requires further ϵ -expansion of the most difficult integrals (calculations are conducted in $D = 4 - 2\epsilon$ dimensions). This is done by assigning a large mass to some internal lines, differentiating with respect to it, and solving a system of differential equations [24].

Apart from this all-hard case, regions with some hard and some soft momenta factorize into diagrams with less than four loops. The most difficult ones have three-loop subgraphs and are evaluated using Ref. [25–27]. Symbolic algebra is done with FORM [28] and GiNaC [29].

Up to the two-gluon order, QCD corrections to the process $b \rightarrow c \ell \bar{\nu}$ are parameterized by

$$\Gamma(b \rightarrow c \ell \bar{\nu}) = \Gamma_0 C_F \left[\frac{X_0}{C_F} + \frac{\alpha_s}{\pi} X_1 + \left(\frac{\alpha_s}{\pi} \right)^2 X_2 + \dots \right],$$

where $\Gamma_0 = \frac{G_F^2 |V_{cb}|^2 m_b^5}{192\pi^3}$ is the tree-level massless result, and

$$\begin{aligned} X_C = & -\frac{1009}{288} + \frac{8\zeta_3}{3} + \frac{77\pi^2}{216} - \frac{5}{4}\pi^2\rho + \left[\frac{145}{3} + \frac{52}{3}\ln\rho - 8\ln^2\rho + \frac{16\pi^2}{3} \right] \rho^2 + \left[\frac{569}{36} + \frac{64}{3}\ln\rho \right] \pi^2\rho^3 + \dots, \\ X_N = & \frac{154927}{10368} - \frac{383\zeta_3}{72} + \frac{95\pi^2}{162} - \frac{53\pi^2\ln 2}{12} + \frac{101\pi^4}{1440} + \left[\frac{539\pi^4}{1080} - \frac{1181\pi^2}{216} - \frac{185}{3}\ln\rho + 22\ln^2\rho - 43\zeta_3 + 4\pi^2\ln 2 - \frac{1537}{16} \right] \rho^2 \\ & + \left[\frac{556}{3} - \frac{1136\ln 2}{3} + \frac{56\pi}{3} - \frac{124}{3}\ln\rho \right] \pi^2\rho^3 + \left[\frac{1777\pi^4}{720} - \frac{23807}{864} + \left(\frac{577}{36} - 39\zeta_3 - \frac{10\pi^4}{3} + 30\pi^2\ln 2 - 15\pi^2 \right) \ln\rho \right. \\ & + (5\pi^2 - 185)\ln^2\rho + 88\ln^3\rho + 4\pi^2\ln^2 2 + 48\text{Li}_4 \frac{1}{2} - \frac{215\pi^2\zeta_3}{6} + \frac{727\pi^2}{48} \\ & \left. - \frac{535\zeta_5}{2} - \frac{615\zeta_3}{4} + 2\ln^4 2 - \frac{13\pi^2\ln 2}{2} \right] \rho^4 + \dots \end{aligned} \quad (3)$$

These corrections significantly depend on ρ near its realistic values: $X_2(0.25) = -6.59$, while $X_2(0.3) = -4.89$.

In addition to the decay rate, corrections to the first two moments in lepton energy $\hat{E}_l = E_l/m_b$, and the hadronic system energy $\hat{E}_h = E_h/m_b$, have been computed. The average is taken over the whole phase space of decay products. These moments are parameterized by $\frac{1}{\Gamma_0} \langle \hat{E}_l^n \rangle \equiv \sum_{j=0}^{\infty} \left(\frac{\alpha_s}{\pi} \right)^j L_j^{(n)}$, and similarly for the moments of \hat{E}_h , described by coefficients $H_j^{(n)}$. Table I shows the second-order corrections $L_2^{(1,2)}$ and $H_2^{(1,2)}$. U_C in Table I is the c -quark contribution to $\Gamma(b \rightarrow u \ell \bar{\nu})$, defined by analogy with X_C of (2). This result is useful, e.g., in $b \rightarrow s \gamma$ studies [31].

α_s is normalized in the modified minimal subtraction ($\overline{\text{MS}}$) scheme at m_b . The tree-level mass-dependent decay rate is $X_0 = 1 - 8\rho^2 - 24\rho^4 \ln\rho + 8\rho^6 - \rho^8$. The one-gluon correction, known exactly [30], has an expansion in ρ starting with

$$\begin{aligned} X_1 = & \frac{25}{8} - \frac{\pi^2}{2} - (34 + 24\ln\rho)\rho^2 + 16\pi^2\rho^3 \\ & - \left(\frac{273}{2} - 36\ln\rho + 72\ln^2\rho + 8\pi^2 \right) \rho^4 + \dots \end{aligned} \quad (1)$$

The second-order correction X_2 is a sum of finite, gauge invariant parts proportional to various color factors,

$$X_2 = C_F X_A + C_A X_N + T_R (X_C + X_H + N_L X_L). \quad (2)$$

X_L , X_C , and X_H denote contributions of c -, b -, and $N_L = 3$ species of massless quarks. $\text{SU}(3)$ color factors are $T_R = \frac{1}{2}$, $C_F = \frac{4}{3}$, $C_A = 3$.

Using techniques described above, expansions of X_A , X_N , X_C , X_L , and X_H are obtained through $\mathcal{O}(\rho^7)$, sufficient for sub-percent accuracy of X_2 for the physical $\rho \approx 0.3$. These expressions being rather lengthy, Table I lists numerical values of the coefficients. X_N and X_C are shown explicitly for the purpose of subsequent discussion,

Our results can be tested by comparing logarithms of the mass ratio with predictions of a renormalization group analysis [19]. That study summed up some of the logarithms to all orders in the coupling constant. When those results are expanded in α_s , three terms can be tested in order α_s^2 : $\rho^2 \ln^2\rho$, $\rho^3 \ln\rho$, and $\rho^4 \ln^3\rho$. They are related to the presence in the tree-level width X_0 of terms ρ^2 and $\rho^4 \ln\rho$, and the absence of ρ^3 there.

Ref. [19] traced the origin of those terms to operators constructed from the b and c quark fields. The logs in terms ρ^2 were found to arise from the running of the c -quark mass, and those in ρ^4 —from a mixing of dimension seven operators. Our results agree with that analysis.

However, terms $\rho^3 \ln\rho$ were attributed in [19] to the running of m_c , with the resulting coefficient that disagrees

TABLE I. Coefficients of $\rho^i \ln^j \rho$ in expansions of the components of X_2 , Eq. (2), and of the moments $\langle \hat{E}_{l,h}^{1,2} \rangle$. The last line shows the analogue of X_C for the decay $b \rightarrow u\ell\bar{\nu}$.

	ρ^0	ρ^1	ρ^2	$\rho^2 \ln \rho$	$\rho^2 \ln^2 \rho$	ρ^3	$\rho^3 \ln \rho$	ρ^4	$\rho^4 \ln \rho$	$\rho^4 \ln^2 \rho$	$\rho^4 \ln^3 \rho$
X_A	3.5588	0	-145.23	-105	-36	-332.91	0	1807.0	-130.61	138.48	-144
X_N	-9.0464	0	-125.74	-61.667	22	-182.54	-407.94	-525.17	-298.36	-135.65	88
X_C	3.2203	-12.337	100.97	17.333	-8	155.99	210.55	112.72	343.28	44	-32
X_L	3.2203	0	26.174	17.333	-8	23.121	105.28	41.907	-13.638	52	-32
X_H	-0.036433	0	-0.30328	0	0	0	0	-8.8409	-8.2856	0	0
$L_2^{(1)}$	-8.1819	-3.4544	-320.89	-185.83	1.25	-581.80	-640.43	-257.71	-1141.8	-245.37	13.111
$L_2^{(2)}$	-3.3806	-1.5353	-160.06	-90.557	0.6	-290.22	-320.21	-820.05	-951.38	-215.13	10.444
$H_2^{(1)}$	-6.1150	-1.8094	-28.786	-46.563	0.41667	4.0514	0	1648.7	914.98	276.28	-2.6667
$H_2^{(2)}$	-1.7910	-0.3838	-3.8498	-6.9712	0.066667	0.90167	0	-645.37	-317.85	-26.783	0
U_C	3.2203	-12.337	47.319	0	0	119.90	105.28	-104.95	120.05	-8	0
	ρ^5	$\rho^5 \ln \rho$	ρ^6	$\rho^6 \ln \rho$	$\rho^6 \ln^2 \rho$	$\rho^6 \ln^3 \rho$	ρ^7	$\rho^7 \ln \rho$			
X_A	-617.86	1579.1	-610.71	91.319	95.033	0	-39.188	301.79			
X_N	263.65	-981.70	540.82	-121.07	-12.203	0	-52.784	-158.92			
X_C	-188.62	164.49	-235.84	87.674	-7.111	0	144.75	177.65			
X_L	-120.76	105.28	22.976	-57.926	24	0	18.799	0			
X_H	0	0	7.5624	-12.813	3.2	0	0	0			
$L_2^{(1)}$	-392.83	-1291.7	1458.5	-757.80	127.03	-8.7037	141.85	-48.398			
$L_2^{(2)}$	-286.40	-1359.2	1080.5	-1028.9	13.504	-16.684	425.16	-211.77			
$H_2^{(1)}$	359.98	1571.0	-1727.4	1184.0	-191.14	29.728	-346.40	110.98			
$H_2^{(2)}$	-358.33	-844.98	78.867	-1134.4	-176.54	-34.547	420.77	-560.13			
U_C	-73.693	0	34.956	1.6	-7.1111	0	-33.839	0			

with our result. We find that they originate from the four-quark operator $\bar{h}_b \Gamma_\mu c \bar{c} \Gamma^\mu h_b$. Here h_b denotes the static field describing a slow quark b . As discussed in Ref. [19], this operator gives rise to terms ρ^3 in the tree-level decay in the case of a *vector* coupling ($\Gamma_\mu = \gamma_\mu$), but does not contribute at the tree-level in the chiral case ($\Gamma_\mu = \gamma_\mu(1 - \gamma_5)$). However, we find that at $\mathcal{O}(\alpha_s)$ it has a finite matrix element as shown in Fig. 1(d), responsible for the cubic mass term in the decay width (1).

At the next loop level, the effects of the coupling constant running and of the anomalous dimension of that operator generate terms $\alpha_s^2 \rho^3 \ln \rho$. Including charm, $N_L + 1$ quarks contribute to the coupling running. Denoting $\beta_0 \equiv \frac{11}{3}C_A - \frac{4}{3}T_R(N_L + 1)$, one finds $-8\beta_0 + \frac{32}{3}T_R - 12C_A = \frac{32}{3}T_R N_L + \frac{64}{3}T_R - \frac{124}{3}C_A$, in agreement with the coefficients of $\pi^2 \rho^3 \ln \rho$ in Eq. (3).

The linear term $-5\pi^2 \rho/4$ in Eq. (3) is noteworthy. As will now be explained, it arises because the on-shell (pole) definition of the b -quark mass has been used. Although not suitable for phenomenology, it has been adopted for the ease of comparisons with Ref. [20] and simplicity of presentation. The linear term m_q/m_b arises from a q -quark ($m_q \ll m_b$) loop inserted in the gluon propagator in Fig. 1(d). Thus it does not depend on the final-state quark mass and equally affects decays $b \rightarrow c\ell\bar{\nu}$ and $b \rightarrow u\ell\bar{\nu}$. Arising from the gluons with momentum $\mathcal{O}(m_q)$ this

effect becomes a problem for the perturbative analysis when q is light, $m_q \leq \Lambda_{\text{QCD}}$. This illustrates how a linear Λ_{QCD}/m_b correction appears even in the total b -quark width when the pole mass is used. If a short-distance mass definition is used, such as the $\overline{\text{MS}}$ mass [32], such terms are absorbed into the lowest-order decay width and are absent in higher-orders of the α_s expansion [33]. Note the factor 5 in the coefficient of the linear term, related to the fifth power of m_b in the width formula.

Where the linear correction really counts is the muon decay width, traditionally expressed using the muon pole mass m_μ . Results for the muon are obtained by setting $C_F = T_R = V_{cb} = 1$ and $C_A = N_L = 0$, and replacing m_b by m_μ and ρ by m_e/m_μ . The previous study of this decay [9] neglected the electron mass m_e . Its effect on the decay rate was assumed to arise from terms $(\frac{\alpha}{\pi})^2 (\frac{m_e}{m_\mu})^2 \ln^2(\frac{m_e}{m_\mu})^2 \simeq 1.5 \times 10^{-8}$. The theoretical error was estimated by taking the coefficient of this term to be 24. Because of the overlooked linear correction the electron mass effect turns out to be larger.

The Fermi constant is determined from the measured muon lifetime using the relation [9] $\frac{1}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3} (1 + \Delta q)$, where Δq describes effects of the finite electron mass and radiative corrections in the limit of the four-fermion contact interaction ($m_\mu \ll M_W$). The latter have been known exactly in the first order in α [34], and in the limit of zero

m_e in order α^2 . Present result extends the knowledge of the α^2 correction to the case of the finite m_e . The extra shift is $\Delta q(m_e) \approx -0.43 \cdot 10^{-6}$, comparable with the expected precision of the MuLan result.

Even though the coefficient of the $(\frac{\alpha}{\pi})^2 (\frac{m_e}{m_\mu})^2 \ln^2(\frac{m_e}{m_\mu})^2$ term is only -11 , less than half the value taken in [9] for the error estimate, the total electron mass effect, dominated by the linear term m_e/m_μ , is larger than expected.

For the decay $b \rightarrow c \ell \bar{\nu}$, we confirm numerical results of [20]. The flexible numerical method developed in that reference enables one to impose phase space cuts. Our approach is complementary. It provides analytical results and gives insight into the small-mass region, important for the muon decay. It also facilitates changes of the scheme used for the heavy-quark masses. Finally, it reveals the structure of logarithms and highlights the relative importance of various operators, improving on the analysis of [19]. But is it possible to combine the analytical approach with cuts on the lepton energy?

According to [20], effects of cuts for the *lepton energy* moments can be modeled with the tree-level distribution [35]. For the cut $E_l > E_{\text{cut}} = 1 \text{ GeV}$ one finds $L_{2,\text{cut}}^{(n)}/L_2^{(n)} \approx L_{1,\text{cut}}^{(n)}/L_1^{(n)} \approx L_{0,\text{cut}}^{(n)}/L_0^{(n)}$. For the rate $L_{2,\text{cut}}^{(0)}$ the error is only -4% , and less than -2% for $L_{2,\text{cut}}^{(1,2)}$, indicating smallness of the hard-gluon radiation.

The relative impact of cuts on the *hadron energy* moments $\langle \hat{E}_h^n \rangle$ depends on the order n of the moment only very weakly [20]. To within 1%, $H_{2,\text{cut}}^{(1)}/H_2^{(1)}$ and $H_{2,\text{cut}}^{(2)}/H_2^{(2)}$ both equal $L_{2,\text{cut}}^{(0)}/L_2^{(0)}$ (and significantly exceed $H_{0,\text{cut}}^{(1,2)}/H_0^{(1,2)}$). First-order corrections behave similarly: $H_{1,\text{cut}}^{(1)}/H_1^{(1)} \approx H_{1,\text{cut}}^{(2)}/H_1^{(2)} \approx L_{1,\text{cut}}^{(0)}/L_1^{(0)}$ within 0.1%. Thus the effect of cuts $E_{\text{cut}} \lesssim 1 \text{ GeV}$ on corrections to $\langle \hat{E}_h^n \rangle$ can be approximated by that on the rate.

Our result for the correction to the width exceeds the earlier estimate based on an interpolation of special kinematic results [13]. The full mass dependence being now known, it is clear that the correction varies strongly as a function of the mass ratio and is close to vanishing near the physical value. This indicates cancellations that were not taken into account in [13]. In particular [20], the non-BLM correction turns out to be about $1.7(\alpha_s/\pi)^2$. While the full phenomenological analysis requires a fit of the rate and moments, we expect this correction to decrease the value of $|V_{cb}|$ determined from inclusive b decays by about 1%, bringing it slightly closer to the exclusive-decay result.

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