Constraints on the Infrared Behavior of the Gluon Propagator in Yang-Mills Theories

A. Cucchieri

Instituto de Física de São Carlos, Universidade de São Paulo, Caixa Postal 369, 13560-970 São Carlos, SP, Brazil

T. Mendes*

DESY–Zeuthen, Platanenallee 6, 15738 Zeuthen, Germany

(Received 21 December 2007; published 18 June 2008)

We present rigorous upper and lower bounds for the zero-momentum gluon propagator $D(0)$ of Yang-Mills theories in terms of the average value of the gluon field. This allows us to perform a controlled extrapolation of lattice data to infinite volume, showing that the infrared limit of the Landau-gauge gluon propagator in $SU(2)$ gauge theory is finite and nonzero in three and in four space-time dimensions. In the two-dimensional case, we find $D(0) = 0$, in agreement with Maas. We suggest an explanation for these results. We note that our discussion is general, although we apply our analysis only to pure gauge theory in the Landau gauge. Simulations have been performed on the IBM supercomputer at the University of São Paulo.

DOI: 10.1103/PhysRevLett.100.241601 PACS numbers: 11.15.Ha, 12.38.Aw, 14.70.Dj

Introduction.—Color confinement is a basic feature of hadron physics that still lacks a clear theoretical understanding. Among several explanations suggested in the literature (see [1] for a recent review), the so-called Landau-gauge Gribov-Zwanziger scenario [2,3] relates gluon confinement to the infrared (IR) suppression of the gluon propagator $D(p^2)$, whereas quark confinement is related to the IR enhancement of the ghost propagator $G(p^2)$. This scenario is supported by several studies using functional methods [4]. In particular, these studies [5–7] predict, for small momenta, a gluon propagator $D(p^2) \propto p^{2(a_D-1)}$ and a ghost propagator $G(p^2) \propto 1/p^{2(1+a_G)}$. The IR exponents a_D and a_G should satisfy the relation $a_D = 2a_G + (4-d)/2$, where d is the space-time dimension and a_G should have a value in the interval $[1/2, 1]$. Clearly, if $a_D > 1$, one has $D(0) = 0$, implying maximal violation of reflection positivity [3]. In the four-dimensional case, one finds [6,7] that $a_G \approx 0.59$ and $a_D = 2a_G$. Similar power behaviors have also been obtained for the various vertex functions of $SU(N_c)$ Yang-Mills theories [7,8]. As a consequence, the running coupling constants from the ghost-gluon, three-gluon, and four-gluon vertices are all finite at zero momentum, displaying a universal (qualitative) behavior [4]. Let us note that a key ingredient of these results is the nonrenormalization of the ghost-gluon vertex, i.e., $\tilde{Z}_1(p^2) = 1$, which has been verified at the nonperturbative level [9] using lattice Monte Carlo simulations.

One should stress, however, that different IR behaviors for the Landau gluon and ghost propagators have also been proposed in the literature. For example, in Ref. [10], the authors find that $D(0)$ is finite and nonzero and that $a_G \approx 0$, with a gluon propagator characterized by a dynamically generated mass. Similar results are obtained in Ref. [11]. On the other hand, in Ref. [12], using Ward-Slavnov-Taylor identities, the authors conclude that $D(p^2)$ should be (probably very weakly) divergent at small momenta and that $a_G = 0$. Recently, Chernodub and Zakharov [13] obtained the relation $2a_D + a_G = 1$ for the 4D IR exponents of gluon and ghost propagators, by considering the contribution of these propagators to thermodynamic quantities of the system, such as pressure and energy density. This result, together with the previous relation between a_D and a_G, implies that $a_D = 2/5$ and $a_G = 1/5$; i.e., the ghost propagator blows up faster than p^{-2} at small momenta, while the gluon propagator diverges as $p^{-6/5}$. Very recently, in Ref. [14], it was shown that by using the Gribov-Zwanziger approach one can also obtain a finite $D(0)$ gluon propagator and $a_G = 0$. Finally, phenomenological tests [15] seem to favor a finite and nonzero $D(0)$.

Numerical studies using Monte Carlo simulations suggest that the gluon propagator is finite at zero momentum [16–20] and that the ghost propagator [16,17,21] is enhanced when compared to the tree-level behavior p^{-2}. Moreover, in 2D and in 3D [16,20] the gluon propagator $D(p^2)$ shows a maximum value for p of a few hundred MeV and decreases as p goes to 0. On the other hand, in 4D, even using lattices with a lattice side of about 10 fm, one does not see a gluon propagator decreasing at small momenta [19]. It has been argued that an IR decreasing gluon propagator can be obtained numerically only when simulations are done on large enough lattice sizes [22]. However, from recent studies in 4D using very large lattice volumes [23–25], one sees that $D(p^2)$ either displays a plateau for momenta $p \lesssim 100$ MeV or gets slightly suppressed at small momenta. Let us note that one of the main problems of the numerical studies of the gluon propagator is the lack of a simple way of extrapolating the data to infinite volume.

Here we discuss the behavior of the gluon propagator at zero momentum. We show that, instead of studying $D(0)$ directly, it is more convenient to consider the quantity

$$M(0) = \frac{1}{d(N_c^2-1)} \sum_{\mu,b} |\tilde{A}_\mu^b(0)|. \qquad (1)$$

In a spin system, this would be equivalent to studying the average absolute value of the components of the magnetization instead of the susceptibility, which is, of course, a much noisier quantity. (Note that, by symmetry, the field components will average to zero if no absolute value is taken.) In order to relate $M(0)$ to $D(0)$, we derive rigorous lower and upper bounds for $D(0)$, which are expressed in terms of $M(0)$. Numerical data are obtained from extensive simulations in two, three, and four dimensions, for the pure $SU(2)$ case, using very large lattices in the scaling region. We show that by using these bounds for $D(0)$ and with present lattice sizes we have clear control over the extrapolation of the data to the infinite-volume limit. We suggest a possible explanation of the results obtained. Finally, we present our conclusions. We note that our discussion concerning the bounds for the gluon propagator is general, although we consider here only the Landau-gauge propagator and pure $SU(2)$ gauge theory. Note also that recent studies [25,26] have verified the analytic prediction that Landau-gauge gluon and ghost propagators in $SU(2)$ and in $SU(3)$ are rather similar. Thus, we expect that the analysis presented here should apply also to the $SU(3)$ case.

Lower and upper bounds for $D(0)$.—As noted in the introduction, interesting lower and upper bounds for the gluon propagator at zero momentum $D(0)$ can be obtained by considering the quantity $M(0)$ defined in Eq. (1), i.e., the average of the absolute value of the components of the gluon field at zero momentum. These components are given by

$$\tilde{A}^{\,b}_{\mu}(0) = \frac{1}{V}\sum_{x} A^{b}_{\mu}(x). \tag{2}$$

In Ref. [3], it was shown that in the Landau and the Coulomb gauge the quantity $M(0)$ should go to zero at least as fast as $1/N$ in the infinite-volume limit, where N is the number of lattice points per direction. This result is simply a consequence of the positivity of the Faddeev-Popov matrix; i.e., it applies to gauge-fixed configurations that belong to the interior of the first Gribov region.

In order to find the lower and upper bounds for $D(0)$, let us consider the inequality

$$\left(\frac{1}{m}\sum_{i=1}^{m} X_i\right)^2 \le \frac{1}{m}\sum_{i=1}^{m} X_i^2, \tag{3}$$

where $\vec{X}$ is a vector with m components X_i. This result simply says that the square of the average of an observable is smaller than or equal to the average of the square of this quantity and is equivalent to the inequality

$$\frac{1}{m}\sum_{i=1}^{m}(X_i - \bar{X})^2 \ge 0, \qquad \bar{X} = \frac{1}{m}\sum_{i=1}^{m} X_i. \tag{4}$$

Note that expression (3) becomes an equality when $X_i =$ constant. We now apply (3) to the vector with $m = d(N_c^2 - 1)$ components $\langle|\tilde{A}^b_\mu(0)|\rangle$. This yields

$$\langle M(0)\rangle^2 \le \frac{1}{d(N_c^2-1)}\sum_{\mu,b}\langle|\tilde{A}^b_\mu(0)|\rangle^2. \tag{5}$$

Then we can apply the same inequality to the Monte Carlo estimate of the average value $\langle|\tilde{A}^b_\mu(0)|\rangle = n^{-1}\sum_c|\tilde{A}^b_{\mu,c}(0)|$, where n is the number of configurations. In this case, we obtain

$$\langle|\tilde{A}^b_\mu(0)|\rangle^2 \le \langle|\tilde{A}^b_\mu(0)|^2\rangle. \tag{6}$$

Thus, by recalling that

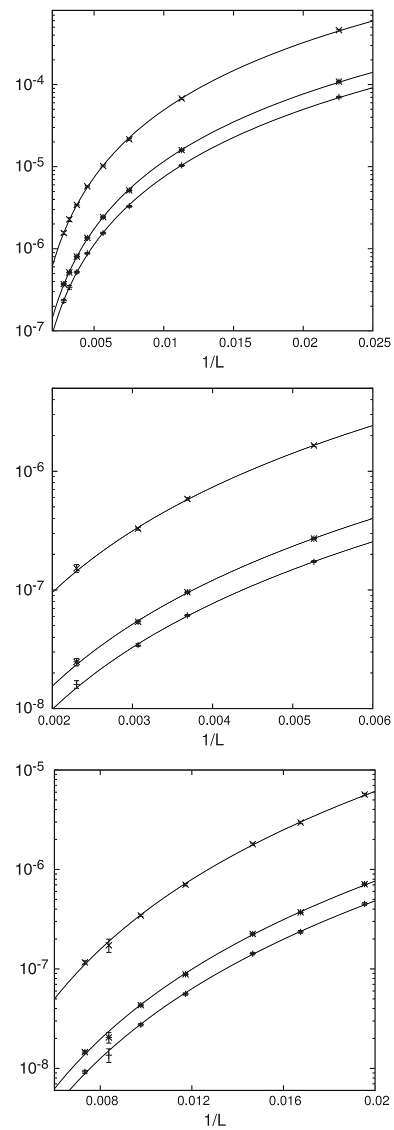

FIG. 1. The square of the quantity $a\langle M(0)\rangle$ and the quantity $a^2 d(N_c^2-1)\langle M(0)^2\rangle$ (both in GeV^{-2}) as a function of the inverse lattice side $1/L$ (GeV) for the 2D case (top), the 3D case (center), and the 4D case (bottom). We also show the data for $a^2 D(0)/V$ (also in GeV^{-2}) and the fit of the data using the parameters reported in Table I. Note that with our notation $M(0)$, $D(0)$, and V are dimensionless quantities, while $L = aV^{1/d}$ is dimensionful.

TABLE I. Fits of $a\langle M(0)\rangle$, $a^2\langle M(0)^2\rangle$, and $a^2 D(0)/V$, respectively, using the *Ansätze* B_l/L^l, B_u/L^u, and B/L^k. Note that B_l, B_u, and B have mass dimensions, respectively, $-l-1$, $-u-2$, and $-k-2$. Note also that, in order to obtain Fig. 1, one should multiply by $d(N_c^2-1)$ the data and the fit related to the fourth and fifth columns of the table. Most of the data used for the fits have a statistical error of the order of 2%–3%. For all fits we have χ^2/d.o.f. ≈ 1.

d	B_l	l	B_u	u	B	k
2	1.48(6)	1.367(8)	2.3(2)	2.72(1)	3.3(2)	2.73(1)
3	1.0(1)	1.48(3)	1.0(3)	2.95(5)	1.5(3)	2.96(4)
4	1.7(1)	1.99(2)	3.1(5)	3.99(4)	4.7(8)	3.99(4)

$$D(0) = \frac{V}{d(N_c^2-1)} \sum_{\mu,b} \langle |\tilde{A}_\mu^b(0)|^2\rangle \tag{7}$$

and using Eqs. (5) and (6), we find that

$$V\langle M(0)\rangle^2 \leq D(0). \tag{8}$$

At the same time we can write the inequality

$$\left\langle \sum_{\mu,b} |\tilde{A}_\mu^b(0)|^2 \right\rangle \leq \left\langle \left[\sum_{\mu,b} |\tilde{A}_\mu^b(0)|\right]^2 \right\rangle. \tag{9}$$

This implies that

$$D(0) \leq V d(N_c^2-1)\langle M(0)^2\rangle. \tag{10}$$

Thus, if $M(0)$ goes to zero as $V^{-\alpha}$, we find that $D(0) \to 0$, $0 < D(0) < +\infty$, or $D(0) \to +\infty$, respectively, if the exponent α is larger than, equal to, or smaller than $1/2$. Finally, let us note that the inequalities (8) and (10) can be immediately extended to the case $D(p^2)$, with $p \neq 0$.

Results.—We have considered several lattice volumes in 2D (at $\beta = 10$, up to a lattice volume $V = 320^2$), in 3D (at $\beta = 3$, up to $V = 320^3$,) and in 4D (at $\beta = 2.2$, up to $V = 128^4$). Details of the simulations will be presented elsewhere [27]. We set the lattice spacing a by considering the input value $\sigma^{1/2} = 0.44$ GeV, which is a typical value for this quantity in the 4D $SU(3)$ case. Note that the lattice volumes 320^2 at $\beta = 10$, 320^3 at $\beta = 3$, and 128^4 at $\beta = 2.2$ correspond, respectively, to $a^2V \approx (70\text{ fm})^2$, $a^3V \approx (85\text{ fm})^3$, and $a^4V \approx (27\text{ fm})^4$. Simulations in 2D have been done on a PC cluster at the IFSC–USP (with 4 Pentium IV 2.8 GHz and 4 Pentium IV 3.0 GHz). Simulations in 3D and in 4D have been done in the 4.5 Tflops IBM supercomputer at USP [28]. The total CPU time was equivalent to about 5.7 (in 3D) and 25.9 days (in 4D) on the whole machine.

We start by considering the quantity $\langle M(0)\rangle$. We find (see Fig. 1 and Table I) that our data extrapolate very well to zero as $1/L^l$, with the values of l given in Table I. Thus, in 3D and in 4D we have $\langle M(0)\rangle \sim 1/V^{1/2}$, implying that $D(0) > 0$. In particular, from our fits we obtain $a^2D(0) \geq (B_l/a^l)^2$. This gives $a^2D(0) \geq 0.4(1)\text{ GeV}^{-2}$ in 3D and $a^2D(0) \geq 2.2(3)\text{ GeV}^{-2}$ in 4D, where we used $a = 1.356\,87\text{ GeV}^{-1}$ in 3D and $a = 1.066\text{ GeV}^{-1}$ in 4D. As for the upper bound (10), by using our fits (see again Fig. 1 and Table I) we have $a^2D(0) \leq d(N_c^2-1)B_u/a^u$, yielding $a^2D(0) \leq 4(1)\text{ GeV}^{-2}$ in 3D and $a^2D(0) \leq 29(5)\text{ GeV}^{-2}$ in 4D. On the other hand, in 2D both the lower and the upper bounds extrapolate to zero, implying that $D(0) = 0$ in agreement with Ref. [16]. Let us note that our bounds in 3D and in 4D are in agreement with the data shown in Figs. 1 and 2 of Ref. [23]. [In the 3D case, compared to the extrapolation reported in Fig. 1 of Ref. [23], one should also include here a factor $\beta = 3.0$, i.e., $1.2(3) \leq a^2D(0) \leq 12(3)\text{ GeV}^{-2}$.] Also note that in the three cases one finds $B_u \approx B_l^2$ and $u \approx 2l$. Indeed, one can check that $\langle M(0)\rangle^2 \lesssim \langle M(0)^2\rangle$, implying that the quantity $M(0)$ is almost the same for all Monte Carlo configurations. More precisely, we verified for the three cases that $\langle M(0)^2\rangle - \langle M(0)\rangle^2$ [i.e., the susceptibility of $M(0)$] goes to zero as $\sim 1/V$ in the infinite-volume limit.

In order to interpret these results, let us first note that, given a Gaussian random variable x with a null mean value and standard deviation σ, the random variable $|x|$ has a mean value (and standard deviation) proportional to σ. In our case, this suggests that the average value of the gluon field at zero momentum $\tilde{A}(0)$ [defined in Eq. (2)] should be zero with a standard deviation of the order of $1/L^c$, with $c \approx l$ (see Table I). This is indeed the case in 2D, 3D, and 4D. [We find, respectively, $c = 1.36(2)$, $1.47(3)$, and $1.97(1)$ for the three cases.] In 3D and in 4D, our results imply that $\sigma \propto 1/\sqrt{V}$. This property is known as *self-averaging* [29] and is the behavior expected for extensive quantities in pure phases, away from phase boundaries. (In our case, the magnetization is not extensive, because we divide by the volume, but the result holds for the relative standard deviation.) More precisely, one talks of strong self-averaging when $\sigma \propto 1/L^c$ and $c = d/2$ and of weak self-averaging when $c < d/2$. Thus, we find strong self-averaging for $M(0)$ in 3D and in 4D and some kind of *over-self-averaging* in 2D, with $c > d/2$. In simpler terms, the gluon propagator may be thought of as the susceptibility associated to the magnetization $M(0)$ [or rather to the quantity defined by Eq. (1) *without* the absolute value, which has zero average]. In 3D and 4D the system has (finite) nonzero susceptibility, while for 2D the susceptibility is zero. We do not have a simple explanation for this latter result. Here we can argue only that the 2D case is probably different since there are no propagating degrees of freedom.

Note that our results in 3D and in 4D imply only that reflection positivity is not *maximally* violated. A clear violation of reflection positivity [27,30] is still obtained in 2D, 3D, and 4D for the $SU(2)$ and $SU(3)$ cases.

Conclusions.—We have shown that the Landau-gauge gluon propagator at zero momentum $D(0)$ is finite and nonzero in 3D and in 4D. At the same time, we find that $D(0) = 0$ in 2D, in agreement with Ref. [16]. These results have been obtained by considering the inequalities in Eqs. (8) and (10), i.e., by studying the ''magnetizationlike'' quantity $M(0)$ instead of the ''susceptibility'' $D(0)$. This allows control of the extrapolation of the data to infinite volume. Moreover, the quantity $D(0)/V$ can be well fitted in this limit as a function of $1/L$. Our results in 3D and in 4D can be explained as a manifestation of strong self-averaging. As mentioned above, a similar analysis may be applied to more general cases and considering also nonzero momenta.

We thank S. Sorella and D. Zwanziger for helpful discussions. We acknowledge partial support from FAPESP and from CNPq. The work of T. M. is supported also by the Alexander von Humboldt Foundation. Most of the simulations reported here have been done on the IBM supercomputer at São Paulo University (FAPESP Grant No. 04/08928-3).

*Permanent address: Instituto de Física de São Carlos, Universidade de São Paulo, C.P. 369, 13560-970 São Carlos, SP, Brazil.

[1] R. Alkofer and J. Greensite, J. Phys. G **34**, S3 (2007).
[2] V. N. Gribov, Nucl. Phys. **B139**, 1 (1978); D. Zwanziger, Nucl. Phys. **B412**, 657 (1994); Y. L. Dokshitzer and D. E. Kharzeev, Annu. Rev. Nucl. Part. Sci. **54**, 487 (2004).
[3] D. Zwanziger, Phys. Lett. B **257**, 168 (1991); Nucl. Phys. **B364**, 127 (1991).
[4] C. S. Fischer, J. Phys. G **32**, R253 (2006).
[5] L. von Smekal, A. Hauck, and R. Alkofer, Ann. Phys. (N.Y.) **267**, 1 (1998) **269**, 182(E) (1998); J. Braun, H. Gies, and J. M. Pawlowski, arXiv:0708.2413.
[6] D. Zwanziger, Phys. Rev. D **65**, 094039 (2002); J. M. Pawlowski, D. F. Litim, S. Nedelko, and L. von Smekal, Phys. Rev. Lett. **93**, 152002 (2004).
[7] C. Lerche and L. von Smekal, Phys. Rev. D **65**, 125006 (2002).
[8] C. S. Fischer and J. M. Pawlowski, Phys. Rev. D **75**, 025012 (2007).
[9] A. Cucchieri, T. Mendes, and A. Mihara, J. High Energy Phys. 12 (2004) 012; E. M. Ilgenfritz *et al.*, Braz. J. Phys. **37**, 193 (2007); A. Cucchieri, A. Maas, and T. Mendes, Phys. Rev. D **74**, 014503 (2006).
[10] A. C. Aguilar and A. A. Natale, J. High Energy Phys. 08 (2004) 057.
[11] M. Frasca, arXiv:0709.2042.
[12] Ph. Boucaud *et al.*, Eur. Phys. J. A **31**, 750 (2007); J. High Energy Phys. 03 (2007) 076.
[13] M. N. Chernodub and V. I. Zakharov, Phys. Rev. Lett. **100**, 222001 (2008).
[14] D. Dudal, S. P. Sorella, N. Vandersickel, and H. Verschelde, Phys. Rev. D **77**, 071501 (2008).
[15] A. A. Natale, Braz. J. Phys. **37**, 306 (2007).
[16] A. Maas, Phys. Rev. D **75**, 116004 (2007).
[17] A. Cucchieri, Nucl. Phys. **B508**, 353 (1997); E. M. Ilgenfritz *et al.*, Braz. J. Phys. **37**, 193 (2007).
[18] D. B. Leinweber, J. I. Skullerud, A. G. Williams, and C. Parrinello, Phys. Rev. D **58**, 031501 (1998); F. D. R. Bonnet *et al.*, Phys. Rev. D **64**, 034501 (2001); S. Furui and H. Nakajima, Phys. Rev. D **69**, 074505 (2004).
[19] A. Cucchieri and T. Mendes, Braz. J. Phys. **37**, 484 (2007).
[20] A. Cucchieri, Phys. Rev. D **60**, 034508 (1999); A. Cucchieri, T. Mendes, and A. R. Taurines, Phys. Rev. D **67**, 091502 (2003).
[21] A. Sternbeck, E. M. Ilgenfritz, M. Müller-Preussker, and A. Schiller, Phys. Rev. D **72**, 014507 (2005); Ph. Boucaud *et al.*, Phys. Rev. D **72**, 114503 (2005); I. L. Bogolubsky, G. Burgio, M. Müller-Preussker, and V. K. Mitrjushkin, Phys. Rev. D **74**, 034503 (2006); A. Cucchieri, A. Maas, and T. Mendes, Phys. Rev. D **74**, 014503 (2006); O. Oliveira and P. J. Silva, Braz. J. Phys. **37**, 201 (2007), http://www.sbfisica.org.br/bjp/; Eur. Phys. J. A **31**, 790 (2007).
[22] C. S. Fischer, R. Alkofer, and H. Reinhardt, Phys. Rev. D **65**, 094008 (2002); C. S. Fischer, B. Gruter, and R. Alkofer, Ann. Phys. (N.Y.) **321**, 1918 (2006); C. S. Fischer, A. Maas, J. M. Pawlowski, and L. von Smekal, Ann. Phys. (N.Y.) **322**, 2916 (2007); C. S. Fischer *et al.*, Proc. Sci., LATTICE2007 (2007) 300.
[23] A. Cucchieri and T. Mendes, Proc. Sci., LATTICE2007 (2007) 297.
[24] I. L. Bogolubsky, E. M. Ilgenfritz, M. Muller-Preussker, and A. Sternbeck, Proc. Sci., LATTICE2007 (2007) 290.
[25] A. Sternbeck, L. von Smekal, D. B. Leinweber, and A. G. Williams, Proc. Sci., LATTICE2007 (2007) 340.
[26] A. Cucchieri, T. Mendes, O. Oliveira, and P. J. Silva, Phys. Rev. D **76**, 114507 (2007).
[27] A. Cucchieri and T. Mendes (to be published).
[28] http://www.usp.br/lcca/IBM.html.
[29] D. P. Landau and K. Binder, *A Guide to Monte Carlo Simulations in Statistical Physics* (Cambridge University Press, Cambridge, England, 2000).
[30] K. Langfeld, H. Reinhardt, and J. Gattnar, Nucl. Phys. **B621**, 131 (2002); A. Cucchieri, T. Mendes, and A. R. Taurines, Phys. Rev. D **71**, 051902 (2005); P. J. Silva and O. Oliveira, Proc. Sci., LAT2006 (2006) 075; A. Sternbeck *et al.*, Proc. Sci., LAT2006 (2006) 076; P. O. Bowman *et al.*, Phys. Rev. D **76**, 094505 (2007).