Hysteresis-Induced Long-Time Tails

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It is shown analytically that the output of the standard model for complex, nonlocal hysteresis, the Preisach model, exhibits long-time tails under quite general conditions. For uncorrelated input signals the exponent of the algebraic output correlation decay is determined solely by the tails of the input and the Preisach density. Correspondingly we identify universality classes leading to identical algebraic tails. These results predict the occurrence of $1/f$ noise for a large class of hysteretic systems.

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Hysteresis is a well-known and very general phenomenon observed in many branches of science [1]. It refers to nonequilibrium scenarios, where even for arbitrarily slow external parameter or field variations a huge number of internal system states are accessible for a given value of the external field. The actual state of the system depends on previous parameter variations, which may have occurred arbitrarily far in the past, and therefore in a nontrivial way on the history of the system. Accordingly, the input-output relations of these systems, in addition to a major hysteresis loop, show a complicated subloop structure as the input is varied. Classical examples for such complex hysteretic behavior with nonlocal memory are magnetic materials, where magnetization and external magnetic field are hysteretically related [2], still a subject of current research due to the recent developments in the nanosciences (see, e.g., [3]). The ubiquitous presence of complex hysteresis was recognized quite early [4] and recent examples range from shape memory alloys [5], piezoelectric materials [6], superconducting systems [7], and porous materials such as soils [8] or foils [9] to consolidated materials [10], and even economic systems [11]. The hysteretic behavior of such systems is often difficult to access from first principles. A phenomenological model, which is successfully applied to the mentioned systems, also in all the works cited above, is the so-called Preisach model [12]. The universal properties of this standard model, its limitations, and various extensions were elaborated in detail in [13]. In the mathematical literature the associated Preisach operator was also investigated extensively [14]. Despite this there do not exist many rigorous results on general characteristic properties of the output time series generated by this model. Especially in view of its internal nonlocal memory, one wonders whether this model is able to exhibit long-time correlations in the output. Often in the context of the mentioned systems spatially extended models, such as the nonequilibrium zero-temperature random field Ising model [15] or models for self-organized criticality (see, e.g., [16]), are invoked to explain long-time tails and $1/f$ noise via avalanches in space. Below it is shown analytically that already the simpler Preisach model, an independent domain model with no spatial degrees of freedom, is able to explain the occurrence of algebraically decaying correlation functions and $1/f$ power spectra.

The Preisach model transforms input time series $x(t)$ into output $y(t)$ by the action of the Preisach operator $y(t) = \iint d\alpha d\beta \mu(\alpha, \beta) s_{\alpha\beta}[x(t)]$. Here $s_{\alpha\beta}[x(t)]$ is the output of an elementary rectangular hysteresis loop, which can take the values $s_{\alpha\beta} = \pm 1$. It is characterized by a threshold value α, where for rising input $x(t)$ the unit switches from -1 to $+1$, and a smaller threshold value β, which determines the external field value, where such a unit switches back from $+1$ to -1 for decreasing input. Thus the output of the Preisach model consists of the weighted superposition of the output of infinitely many different delayed relays. The weighting function $\mu(\alpha, \beta)$, the so-called Preisach density, non-negative for $\alpha > \beta$ and zero elsewhere, is characteristic for a given system and can be easily obtained from experimental data [13]. The internal state of the Preisach model at time t is given by a piecewise constant function in the α-β plane, the Preisach plane, which separates units in state $s_{\alpha\beta} = +1$ from those with $s_{\alpha\beta} = -1$. This internal state changes dynamically in dependence on the input and it is the place where information about previous input extrema is stored or deleted [13]. In the following we consider input sequences in discrete time $t = 0, 1, 2, \ldots$ with elements $\{x(t)\}$. For such time series the action of the Preisach operator is uniquely defined at all integer steps if it is understood that between the discrete time instants the input is supplemented by a continuous time component varying monotonically from $x(t)$ to $x(t+1)$. The exact results presented below are all obtained for $\{x(t)\}$ being a stochastic process consisting of independent, identically distributed (IID) random variables with density $\rho(x)$. The dynamics of the mean output under such assumptions was already investigated analytically in the context of viscosity modeling [17]. For output correlation functions no exact results have been found previously, although first steps were established for input from the more complicated Ornstein-Uhlenbeck process [18].

We consider the stationary two-point output correlation function $C(t) = \langle y(0)y(t)\rangle - \langle y(0)\rangle^2$. Here we can only sketch the derivation of our results; details can be found in [19]. First note that we can express $C(t)$ in terms of the cross correlation function of the output of two elementary relays with thresholds (α, β) and (α', β'), respectively, by use of $\langle y(0)y(t)\rangle = \iint d\alpha d\beta \mu(\alpha, \beta) \iint d\alpha' d\beta' \mu(\alpha', \beta') \times \langle s_{\alpha\beta}[x(0)] s_{\alpha'\beta'}[x(t)]\rangle$. The state of the two relays at some discrete time t is described by the state vector $\mathbf{S}(t) = (s_{\alpha\beta}(t), s_{\alpha'\beta'}(t)) \in \{(1, 1), (1, -1), (-1, 1), (-1, -1)\} \equiv \{\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \mathbf{S}_4\}$. It is easy to see that for IID random input the output of the two relays is described by a 4-state Markov process: It is only the current state $\mathbf{S}(t)$ of the 2-relay system which determines the probability for the next state $\mathbf{S}(t+1)$. This defines the (4×4)-transition matrix $\mathbf{P}$ with elements $(\mathbf{P})_{ij} = p(\mathbf{S}_j | \mathbf{S}_i) = p(\mathbf{S}(t+1) = \mathbf{S}_j | \mathbf{S}(t) = \mathbf{S}_i)$, the conditional probability for being at time $t+1$ in state $\mathbf{S}_j$, provided the system was in state $\mathbf{S}_i$ at time t. Its general form is found to be $\mathbf{P} = \langle \mathbf{P}_{\alpha\beta}(x) \otimes \mathbf{P}_{\alpha'\beta'}(x)\rangle$ with

$$\mathbf{P}_{\alpha\beta}(x) = \begin{pmatrix} \theta(x-\beta) & \theta(\beta - x) \\ \theta(x-\alpha) & \theta(\alpha - x) \end{pmatrix},$$

where $\theta(x)$ is the Heaviside step function. Actually, depending on the mutual order relation between the four threshold parameters β', α', β, and α, one finds six different explicit forms $\mathbf{P}_{\mathrm{I}}, \ldots, \mathbf{P}_{\mathrm{VI}}$ for $\mathbf{P}$. As an example, for the ordering I: $\beta' < \alpha' < \beta < \alpha$, the transition probability $(\mathbf{P}_{\mathrm{I}})_{13}$ is given by $(\mathbf{P}_{\mathrm{I}})_{13} = p((-1, 1)|(1, 1)) = \langle \theta(\beta - x)\theta(x - \beta')\rangle = \int_{\beta'}^{\beta} \rho(x) dx$. This means that, in case both relays are in the upper state $s = 1$ at time t, the one with thresholds α and β flips to $s = -1$ and the other one does not, only if the random input variable $x(t+1)$ falls into the interval $[\beta', \beta]$. Similarly all elements of $\mathbf{P}$ in the six parameter regimes I,. . .,VI can be determined. Thus the calculation of the stationary cross correlation function $\langle s_{\alpha\beta}[x(0)] s_{\alpha'\beta'}[x(t)]\rangle$ amounts to the determination of a correlation function for a discrete Markov process over four states, which by using the spectral decomposition of $\mathbf{P}$ is a straightforward task. It turns out [19] that in all parameter regimes $\langle s_{\alpha\beta}[x(0)] s_{\alpha'\beta'}[x(t)]\rangle$ shows a simple exponential decay to its asymptotic value $C_{\alpha\beta,\alpha'\beta'}(t) = \langle s_{\alpha\beta}[x(0)] s_{\alpha'\beta'}[x(t)]\rangle - \langle s_{\alpha\beta}\rangle\langle s_{\alpha'\beta'}\rangle = C_{\alpha\beta,\alpha'\beta'}(0) \times F(\beta', \alpha')^t$, where $F(\beta', \alpha') \equiv \int_{\beta'}^{\alpha'} \rho(x) dx$ is the relevant eigenvalue of $\mathbf{P}$. The amplitude $C_{\alpha\beta,\alpha'\beta'}(0)$ is also known exactly, but takes different forms in the parameter regimes I,. . .,VI. This has to be taken into account in calculating the desired correlation function via

$$C(t) = \iint d\alpha d\beta \mu(\alpha, \beta) \times \iint d\alpha' d\beta' \mu(\alpha', \beta') C_{\alpha\beta,\alpha'\beta'}(0) F(\beta', \alpha')^t. \quad (1)$$

We see that the correlation function $C(t)$ can be regarded as a superposition of infinitely many exponentially decaying contributions. Note that due to the normalization of the input density $\rho(x)$ the eigenvalue $F(\beta', \alpha')$ can get arbitrary close to the value $\lambda = 1$ as β' and α' vary in the integral (1). Thus there exists the possibility of a nontrivial, nonexponential decay of $C(t)$. That this is indeed a common scenario will be shown first for the simpler situation, where the elementary loops can be assumed to be symmetric $\alpha = -\beta$, implying a Preisach density of the form $\mu(\alpha, \beta) = \mu(\alpha)\delta(\alpha + \beta)$ with $\alpha > 0$. This physically important case allows for a rather complete understanding of the origin of long-time tails in hysteretic systems [19]. It is advantageous to consider the Z-transformed correlation function $\tilde{C}(z) = \sum_{t=0}^{\infty} C(t) z^{-t}$, which by use of the special form of $\mu(\alpha, \beta)$ and the explicit form of $C_{\alpha\beta,\alpha'\beta'}(0)$ can be represented as $\tilde{C}(z) = \int_0^\infty d\alpha \int_\alpha^\infty d\alpha' \mu(\alpha)\mu(\alpha') \times \frac{4F(-\infty,-\alpha')F(\alpha',\infty)}{[1-F(-\alpha,\alpha)][1-F(-\alpha',\alpha')]} \frac{z}{z-F(-\alpha',\alpha')} + \int_0^\infty d\alpha \int_0^\alpha d\alpha' \mu(\alpha) \times \mu(\alpha') \frac{4F(-\infty,-\alpha)F(\alpha,\infty)}{[1-F(-\alpha,\alpha)][1-F(-\alpha',\alpha')]} \frac{z}{z-F(-\alpha',\alpha')}$. A further simplification is achieved for symmetric input densities $\rho(x) = \rho(-x)$. With the substitution $u \equiv 2F(\alpha, \infty)$ and $v \equiv 2F(\alpha', \infty)$ one obtains

$$\tilde{C}(z) = \int_0^1 du \int_0^u dv \tilde{\mu}(u)\tilde{\mu}(v) \frac{v}{u} \frac{z}{z-1+v} + \int_0^1 du \int_u^1 dv \tilde{\mu}(u)\tilde{\mu}(v) \frac{u}{v} \frac{z}{z-1+v}, \quad (2)$$

where $\tilde{\mu}(u)$ is an effective Preisach density

$$\tilde{\mu}(u) \equiv \frac{\mu(\alpha(u))}{2\rho(\alpha(u))}, \quad (3)$$

with $\alpha(u)$ the inverse of the monotonically decreasing function $u(\alpha) = 2F(\alpha, \infty)$. For certain combinations $\{\rho(x), \mu(\alpha)\}$ of input and Preisach density, respectively, $\tilde{C}(z)$ can be calculated exactly. As an example we consider the case where both distributions are nonzero on a finite interval. Take $\rho(x) = \frac{\nu}{2}(1 - |x|)^{\nu-1}$ for $|x| \leq 1$, and $\mu(\alpha) = \nu'(1-\alpha)^{\nu'-1}$ for $0 \leq \alpha \leq 1$, with $\nu, \nu' > 0$. In this case the effective density takes the simple form

$$\tilde{\mu}(u) = \gamma u^{-1+\gamma}, \quad (4)$$

with $\gamma = \nu'/\nu > 0$. For $\gamma \neq 1$ the result for $\tilde{C}(z)$ can be expressed with hypergeometric functions as $\tilde{C}(z) = \frac{z}{z-1} \times \frac{\gamma}{1-\gamma^2}[F(1, 2\gamma; 2\gamma+1; \frac{1}{1-z}) - \gamma F(1, \gamma+1; \gamma+2; \frac{1}{1-z})]$, and for $\gamma = 1$ we obtain $\tilde{C}(z) = \frac{3}{2} z + \frac{z}{2}(z-1)\ln(\frac{z-1}{z}) + z(z-1)\,\mathrm{Li}_2(\frac{1}{1-z})$, where $\mathrm{Li}_2(z)$ is the Euler dilogarithm. The long-time behavior of $C(t)$ is determined by the behavior of $\tilde{C}(z)$ near $z = 1$. Therefore we calculated the asymptotic behavior of $\tilde{C}(z \to 1)$. It turns out that $\tilde{C}(z)$ is nonanalytic at $z \to 1$ with degree n, where n is given by

$$n = \lceil \min(\gamma, 2\gamma - 1) \rceil, \quad (5)$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. The degree of nonanalyticity of a function $f(x)$ at x_0 is defined as the smallest integer n for which $f^{(n)}(x_0)$, the nth derivative of f at x_0, is discontinuous. If one is interested only in the degree of nonanalyticity and in the origin of Eq. (5) one can easily check directly for the divergence of the integrals for $\tilde{C}(z=1)$ in Eq. (2) and its derivatives. From the full asymptotic behavior of $\tilde{C}^{(n)}(z\to 1)$ we obtained the exact long-time behavior of $C(t)$ by applying Karamata's Tauberian theorem for power series [20] to $\tilde{C}^{(n)}(z)$. One finds the following asymptotic behavior for the correlation functions

$$C(t) \sim \begin{cases} \frac{\gamma}{1-\gamma^2}\Gamma(2\gamma+1)t^{-2\gamma} & \text{for } 0<\gamma<1 \\ t^{-2}\ln t & \text{for } \gamma=1 \\ \frac{\gamma^2}{\gamma^2-1}\Gamma(\gamma+2)t^{-1-\gamma} & \text{for } 1<\gamma<\infty. \end{cases} \quad (6)$$

The spectral density is given by $S(\omega)=2\,\mathrm{Re}[\tilde{C}(z=e^{i\omega})]-C(t=0)$, with $C(t=0)=\lim_{z\to\infty}\tilde{C}(z)=\frac{\gamma}{\gamma+1}$. The degree of nonanalyticity n of $S(\omega)$ at $\omega=0$ is the same as the one of $\tilde{C}(z)$ at $z=1$. Correspondingly the long-time tails of $C(t)$ are reflected in a divergence of $S(\omega)$ or its nth derivative $S^{(n)}(\omega)$ for $\omega\to 0$. From the asymptotic expansion of $\tilde{C}(z)$ for $z\to 1$ we find depending on γ the following nonanalytic behavior of $S(\omega)$ for $\omega\to 0$:

$$\begin{aligned} S(\omega) &\sim \begin{cases} \frac{\gamma^2}{1-\gamma^2}\frac{2\pi}{\cos(\pi\gamma)}|\omega|^{-1+2\gamma} & \text{for } 0<\gamma<1/2 \\ -\frac{4}{3}\ln|\omega| & \text{for } \gamma=1/2 \end{cases} \\ S^{(1)}(\omega) &\sim \begin{cases} \frac{\gamma^2}{1-\gamma^2}\frac{2\pi}{\cos(\pi\gamma)}(2\gamma-1)|\omega|^{-2+2\gamma}\,\mathrm{sgn}\omega & \text{for } 1/2<\gamma<1 \\ \pi\log|\omega|\,\mathrm{sgn}\omega & \text{for } \gamma=1 \end{cases} \\ S^{(n)}(\omega) &\sim \begin{cases} \frac{\gamma}{1-\gamma}\frac{\pi}{\sin(\pi\gamma/2)}\frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-n)}|\omega|^{-n+\gamma}(\mathrm{sgn}\omega)^n & \text{for } n-1<\gamma<n,\ n=2,3,\ldots \\ \frac{\gamma^2}{\gamma^2-1}\Gamma(n+2)i^n\ln\frac{1}{1-\exp(i\omega)}+\text{c.c.} & \text{for } \gamma=n,\ n=2,3,\ldots. \end{cases} \end{aligned} \quad (7)$$

The last line implies that for γ an even integer n the nth derivative $S^{(n)}(\omega)$ of the spectrum diverges logarithmically, whereas for odd n it exhibits a discontinuity. One should emphasize that the result for the input and Preisach density combinations with $0<\gamma<1/2$ implies a new mechanism for the generation of $1/f$ noise. We also note that the superposition of infinitely many exponentially decaying contributions leading to the long-time tails and to $1/f$ noise bears formally in frequency space some similarity to the superposition of Lorentzians as discussed, e.g., in [21]. The difference lies in the more complicated form of the weighting functions, which in addition are nontrivially related to the properties of the input and the distribution of elementary hysteresis relays. In the latter respect an important point is that the results of Eqs. (6) and (7) are not only valid for the above densities with finite support. One easily checks that the following algebraically or exponentially decaying input and Preisach density pairs $\{\rho(x),\mu(\alpha)\}=\{\frac{\nu}{2}(1+|x|)^{-\nu-1},\nu'(1+\alpha)^{-\nu'-1}\}$ and $\{\rho(x),\mu(\alpha)\}=\{\frac{\nu}{2}\exp(-\nu|x|),\nu'\exp(-\nu'\alpha)\}$ lead via Eq. (3) exactly to the same effective Preisach density $\tilde{\mu}(u)$ of Eq. (4), again with $\gamma=\nu'/\nu>0$. Consequently, the above given exact results for $\tilde{C}(z)$ and the asymptotics given by Eqs. (6) and (7) hold also in these cases. Actually, by analyzing the alternative forms of the definition, Eq. (3), namely the transformation law $\tilde{\mu}(u)=\int_0^\infty\mu(\alpha)\delta(u-u(\alpha))d\alpha$ with $u(\alpha)=2\int_\alpha^\infty\rho(x)dx$, or its local form $\tilde{\mu}(u)du=-\mu(\alpha)d\alpha$, one sees that there exist infinitely many pairs $\{\rho(x),\mu(\alpha)\}$ leading exactly to the same $\tilde{\mu}(u)$. The asymptotic results for the long-time tails, Eq. (6), or for the spectrum, (7), hold for an even wider class of systems. From Eq. (2) one sees that the long-time behavior, corresponding to $z\to 1$, is determined solely by the behavior of $\tilde{\mu}(u)$ for $u\to 0$. Therefore only the tails of $\rho(x)$ and $\mu(\alpha)$ determine the form of the long-time tails in $C(t)$. In addition, one may allow for logarithmic corrections in the behavior of $\tilde{\mu}(u\to 0)$ leading basically to the same long-time tails as without these corrections. For instance, the combination of two Gaussians with variance σ^2 and σ'^2 for input and Preisach density, respectively, gives again an effective Preisach density of the form $\tilde{\mu}(u)\sim\gamma u^{-1+\gamma}$, with $\gamma=\sigma^2/\sigma'^2$, but now with logarithmic corrections. Not every combination $\{\rho(x),\mu(\alpha)\}$, however, gives rise to long-time tails. Choose for instance the input density $\rho(x)$ from the class of algebraically decaying functions and assume the Preisach density $\mu(\alpha)$ to be exponentially decaying. In this case we cannot calculate $\tilde{C}(z)$ exactly; one verifies, however, that such a situation corresponds to the limit $\gamma\to\infty$ in the effective Preisach density, Eq. (4). Correspondingly, the degree of nonanalyticity of $\tilde{C}(z=1)$ is infinite, $\tilde{C}(z)$ is analytic. Therefore no long-time tails occur in this case, but output correlations $C(t)$ decay faster than any power of t. The same holds for any other combination where the input density belongs to a class of broader functions than the Preisach density, e.g., $\rho(x)$ exponentially decaying, $\mu(\alpha)$ Gaussian, etc. The criterion is that all derivatives of $\tilde{\mu}(u)$ vanish at $u=0$. In the reversed case, "narrow" input and "broad" Preisach density, the other extreme degree of nonanalyticity $n=0$ and correspondingly $1/f$ noise in the output signal is obtained.

We now briefly return to the original problem, where no assumptions on the symmetry of the elementary relays or the input density are made. This general case is considerably more complicated. Using the explicit form of the amplitudes $C_{\alpha\beta,\alpha'\beta'}(0)$ in Eq. (1) the general form of the Z transform of $C(t)$ can be written as

$\tilde{C}(z) = \int_0^1 du_1 \int_0^{1-u_1} du_2 \tilde{\mu}(u_1, u_2)[\int_{1-u_2}^1 dv_1 \int_0^{1-v_1} dv_2 \times \tilde{\mu}(v_1, v_2) \frac{4u_1 v_2}{(u_1+u_2)(v_1+v_2)} \frac{z}{z-1+v_1+v_2} + \int_{u_1}^{1-u_2} dv_1 \int_0^{u_2} dv_2 \times \tilde{\mu}(v_1, v_2) \frac{4u_1 v_2}{(u_1+u_2)(v_1+v_2)} \frac{z}{z-1+v_1+v_2} + \int_0^{u_1} dv_1 \int_0^{u_2} dv_2 \times \tilde{\mu}(v_1, v_2) \frac{4v_1 v_2}{(u_1+u_2)(v_1+v_2)} \frac{z}{z-1+v_1+v_2} + \int_{u_1}^{1-u_2} dv_1 \int_{u_2}^{1-v_1} dv_2 \times \tilde{\mu}(v_1, v_2) \frac{4u_1 u_2}{(u_1+u_2)(v_1+v_2)} \frac{z}{z-1+v_1+v_2} + \int_0^{u_1} dv_1 \int_{u_2}^{1-u_1} dv_2 \times \tilde{\mu}(v_1, v_2) \frac{4v_1 u_2}{(u_1+u_2)(v_1+v_2)} \frac{z}{z-1+v_1+v_2} + \int_0^{u_1} dv_1 \int_{1-u_1}^{1-v_1} dv_2 \times \tilde{\mu}(v_1, v_2) \frac{4v_1 u_2}{(u_1+u_2)(v_1+v_2)} \frac{z}{z-1+v_1+v_2}]$. A change of variables $u_1 = F(\alpha, \infty)$, $u_2 = F(-\infty, \beta)$, $v_1 = F(\alpha', \infty)$, and $v_2 = F(-\infty, \beta')$ has also been applied. The transformation $(\alpha, \beta) \to (u_1, u_2)$ maps the Preisach half-plane $\alpha > \beta$ to the triangle $u_1 > 0$, $u_2 > 0$, $u_1 + u_2 < 1$. The interesting point is here that again $\tilde{C}(z)$ depends only on an effective density given by $\tilde{\mu}(u_1, u_2) \equiv \frac{\mu(\alpha(u_1), \beta(u_2))}{\rho(\alpha(u_1))\rho(\beta(u_2))}$ and not on the Preisach and input density separately. Similar to the symmetric case this implies that there exist large classes of input and Preisach density combinations leading to the same decay of the output correlations. In general, however, these integrals cannot be computed analytically. We succeeded only for the simplest case $\tilde{\mu}(u_1, u_2) = 1$, i.e., for the case of a constant input and Preisach density and other combinations in this class [19]. We give here only the result showing that long-time tails occur also in this more general case. One finds a logarithmic divergence near $z = 1$ of the second derivative $\tilde{C}^{(2)}(z) \sim -\frac{16}{3} \log(z - 1)$, which implies by application of Karamata's Tauberian theorem an asymptotic correlation decay as $C(t) \sim \frac{16}{3} t^{-3}$. That this long-time tail is not an exceptional case can be seen from the nonanalytic behavior of $\tilde{C}(z)$ at $z = 1$. Indeed, for an effective Preisach density of the asymptotic form $\tilde{\mu}(u_1, u_2) \sim c u_1^{-1+\gamma_1} u_2^{-1+\gamma_2}$ with $\gamma_1 > 0$, $\gamma_2 > 0$, one can show by considering appropriate subregions of the above integrals that $\tilde{C}(z = 1)$ diverges for $\gamma_1 + \gamma_2 < 1/2$. This sufficient condition implies that in a whole parameter region and the corresponding classes of input and Preisach densities the output shows $1/f$ noise and a correlation decay slower than t^{-1}.

In summary, we have shown analytically that the Preisach model, the most prominent and simplest model for complex hysteresis [1], is able to transform uncorrelated input into output with long-time correlations under very general circumstances. The form of the long-time tails depends solely on the tails of the input and Preisach density as coded in the near-zero behavior of an effective Preisach density. This behavior also determines the universality classes resulting in a given exponent of the long-time tail or in $1/f$ noise in the output signal. The mechanism is formally similar to the superposition of Lorentzians discussed in [21]; the more complicated weighting functions, however, are nontrivially related to the properties of the input and the distribution of elementary hysteresis relays. Interestingly, a generalization of the Preisach model, which includes thermal relaxation processes, also predicts $1/f$ noise [22]. This result, however, is found by invoking linear response theory valid in the fully relaxed limit, and is thus complementary to our findings for the standard Preisach model, which does not take into account thermal relaxation processes. Therefore it will be of great interest to explore the regimes between these two extreme cases, where anomalous low frequency behavior is to be expected, correspondingly.

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