Exact Formula of the Distribution of Schmidt Eigenvalues for Dynamical Formation of Entanglement in Quantum Chaos

Hiroto Kubotani*
Institute of Physics, Faculty of Engineering, Kanagawa University, Yokohama 221-8686, Japan

Satoshi Adachi†
Department of Physics, Faculty of Science, Tokyo Institute of Technology, Meguro 152-8550, Japan

Mikito Toda‡
Department of Physics, Faculty of Science, Nara Womens University, Nara 630-8506, Japan
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The exact formula of the one-level distribution of the Schmidt eigenvalues is obtained for dynamical formation of entanglement in quantum chaos. The formula is based on the random matrix theory of the fixed-trace ensemble, and is derived using the theory of the holonomic system of differential equations. We confirm that the formula describes the universality of the distribution of the Schmidt eigenvalues in quantum chaos.

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Quantum entanglement plays a key role in various aspects of quantum mechanics. Nonseparability in entangled states is a specific feature of quantum mechanics [1–3]. Entanglement is also an indispensable resource in quantum computation and quantum communication [4]. In statistical mechanics of quantum systems, entanglement of a system with its surrounding environment is supposed to be the origin of irreversibility and decoherence [5,6]. In particular, dynamical formation of entanglement by quantum chaos plays an important role for the origin of statistical mechanics [7,8]. Thus, it has been an active research area these days [9,10].

Given a quantum state of a two-particle system $|\Phi\rangle$, we can express it by the Schmidt decomposition $|\Phi\rangle = \sum_{i=1}^{N} d_i |i\rangle_1 \otimes |i\rangle_2$, where d_i $(i = 1, \dots, N)$ are non-negative real and satisfy the normalization condition $\sum_{i=1}^{N} d_i^2 = 1$, and are called the Schmidt eigenvalues. Here, we assume for simplicity that the Hilbert spaces of the two subsystems are of equal dimension N. It is known that the entropy $S = -\sum_{i=1}^{N} d_i^2 \log d_i^2$ characterizes the degree of entanglement, and that the Schmidt number $K = 1/\sum_{i=1}^{N} d_i^4$ represents the number of effective degrees of freedom which contribute to entanglement. They play an important role in quantum optics and quantum cryptography [11,12]. The study of the Schmidt eigenvalues is crucial for understanding and utilizing entanglement.

Recently, we have found universality for the statistical properties of the Schmidt eigenvalues in dynamical entanglement involving quantum chaos [13]; the distribution of the low-lying Schmidt eigenvalues near zero (the hard edge) is described well by the random matrix theory [14] of the Laguerre unitary ensemble [15,16]. However, we also have found serious deviation from the ensemble for those eigenvalues near the largest one (the soft edge). This is because the Laguerre ensemble breaks the normalization condition. Thus, the question remains if universality exists over the whole range of the Schmidt eigenvalues. In particular, it is crucial to investigate how the larger Schmidt eigenvalues are distributed, since they dominate the statistical properties of entanglement.

In this Letter, we analytically derive the exact expression of the one-level distribution of the Schmidt eigenvalues for the fixed-trace ensemble [17] using the theory of the holonomic system of differential equations [18]. We find that the expression well describes the one-level distribution of the Schmidt eigenvalues over the whole range for a coupled quantum system exhibiting chaos. This confirms the existence of universality for the distribution of the Schmidt eigenvalues in quantum chaos.

First, we construct a random matrix theory for an arbitrary square complex matrix W. Suppose we impose the following two conditions (i) and (ii) for the probability distribution $P(\{W_{ij}\})$ of its matrix elements W_{ij}: (i) W_{ij} are independent random variables, i.e., $P(\{W_{ij}\}) = \prod_{ij} P_{ij}(W_{ij})$, and (ii) $P(\{W_{ij}\})$ is invariant under the local unitary transformations, i.e., $P(\{W_{ij}\}) = P(\{W'_{ij}\})$ for $W' = UWV^{-1}$, where U and V are arbitrary unitary matrices. Note that the condition (i) is not consistent with the normalization condition $\sum_{i,j=1}^{N} |W_{ij}|^2 = 1$. Then the probability of the Schmidt eigenvalues d_i of the random matrix W is shown to be represented by the distribution for the Laguerre unitary ensemble

$$P^{(\mathrm{L})}(\{\varepsilon_i\}) = C \prod_{1 \le i < j \le N} (\varepsilon_i - \varepsilon_j)^2 e^{-\sum_i \varepsilon_i}, \tag{1}$$

where the new variables $\varepsilon_i \equiv N^2 d_i^2$ take the values from 0 to ∞, and C is a constant [13]. The one-level distribution for the Laguerre ensemble is obtained by integrating Eq. (1) with respect to the $(N-1)$ variables,

$\varepsilon_i (i = 2, \ldots, N)$, as follows

$$P^{(L)}_{1,N}(\varepsilon) = e^{-\varepsilon} \sum_{k=0}^{N-1} \mathcal{L}_k(\varepsilon)^2/N, \tag{2}$$

where $\mathcal{L}_k(\varepsilon)$ is the kth Laguerre polynomial [16].

However, in order to consider the Schmidt eigenvalues d_i for a wave function represented by a random matrix W, we need to impose the normalization condition $\sum_{i=1}^N d_i^2 = 1$. Then, we are led to the probability distribution for the fixed-trace ensemble

$$P^{(F)}(x_1, \ldots, x_N) \equiv C' \delta\Big(1 - \sum_{i=1}^N x_i\Big) P^{(L)}(\{\varepsilon_i\})|_{\varepsilon_i = N^2 x_i}, \tag{3}$$

where $x_i \equiv d_i^2$, and C' is constant, and the delta function represents the normalization condition. For the fixed-trace ensemble, the one-level distribution is

$$P^{(F)}_{1,N}(x) \equiv \int \cdots \int_{[0,1]^{N-1}} dx_2 \ldots dx_N P^{(F)}(x, x_2, \ldots, x_N). \tag{4}$$

We can reduce Eq. (4) to a simple analytical formula [19]

$$P^{(F)}_{1,N}(x) = (N^2 - 1)(1 - x)^{N(N-2)} Q_N(x), \tag{5}$$

where $Q_n(x)$ is a polynomial of $2n - 2$ degree,

$$Q_n(x) = \sum_{j=0}^{2n-2} \frac{(2 - n^2)_j}{(2)_j} A(n - 1, j) x^j (1 - x)^{2n-2-j}. \tag{6}$$

Here, $(a)_j \equiv a(a + 1) \ldots (a + j - 1)$, and for an integer k

$$A(n, 2k) \equiv \frac{(-n)_{2k}}{(\frac{1}{2})_k k!} {}_3F_2(-k, -k + 1/2, n + 2; n + 1 - 2k, 1; 1) \qquad (0 \le 2k \le n) \tag{7}$$

$$\equiv \frac{(2k + 1)!}{(n + 1)[(2k - n)!]^2 (n - k)! (\frac{1}{2})_{n-k}} \times {}_3F_2(-n + k, -n + k + 1/2, 2k + 2; 2k - n + 1, 2k - n + 1; 1) \qquad (n \le 2k \le 2n), \tag{8}$$

$$A(n, 2k + 1) \equiv -\frac{2(-n)_{2k+1}}{(\frac{3}{2})_k k!} {}_3F_2(-k, -k - 1/2, n + 2; n - 2k, 1; 1) \qquad (0 \le 2k + 1 \le n) \tag{9}$$

$$\equiv \frac{2(2k + 2)!}{(n + 1)[(2k + 1 - n)!]^2 (n - k - 1)! (\frac{3}{2})_{n-k-1}} \times {}_3F_2(-n + k + 1, -n + k + 1/2, 2k + 3; 2k - n + 2, 2k - n + 2; 1) \qquad (n \le 2k + 1 \le 2n), \tag{10}$$

where ${}_3F_2$ is a generalized hypergeometric function.

Here, we explain briefly how Eqs. (5)–(10) are obtained; for full details see the forthcoming paper [19]. We utilize the result obtained recently by Kaneko [18]. He extended the famous Selberg integral [16] by inserting further a hypergeometric type weight factor in the integrand and developed a theory of the holonomic system of differential equations to evaluate analytically the extended integral. In the case of our interest, the Selberg-Kaneko integral is expressed in terms of the Appell's hypergeometric function. By the confluence of a singular point in the original Selberg-Kaneko integral, we introduce the confluent Selberg-Kaneko integral. Then, we further apply the Fourier transformation to it to obtain the simplex type Selberg-Kaneko integral. Equation (4) is expressed in terms of this simplex type Selberg-Kaneko integral. The calculation requires the evaluation of nested hypergeometric series and is lengthy.

Next, we study the dynamical evolution of a system composed of two kicked tops [14,20]. The kicked top is a typical model where (i) the phase space volume is finite and (ii) the classical dynamics changes from regular to chaos as we vary the parameter. We numerically estimate the one-level distribution of the Schmidt eigenvalues for the system, and compare it with the analytical expression Eq. (5). The Hamiltonian of the system is

$$H_T = H_1 + H_2 + H_{12}, \tag{11}$$

where $H_i (i = 1, 2)$ are the Hamiltonians of the kicked tops, and H_{12} is the interaction Hamiltonian

$$H_i = \frac{\pi}{2} J_{y_i} + \frac{k_i}{2j_i} J_{z_i}^2 \sum_{n=1}^{\infty} \delta(t - n) \quad (i = 1, 2), \tag{12}$$

$$H_{12} = \frac{c}{\sqrt{j_1 j_2}} J_{z_1} J_{z_2} \sum_{n=1}^{\infty} \delta(t - n), \tag{13}$$

respectively. The parameters $k_i (i = 1, 2)$ and c are the strengths of the kicks and the interaction, respectively. Here J_{x_i}, J_{y_i}, and J_{z_i} are the components of the angular momenta of the tops, and $j_i (i = 1, 2)$ are the quantum numbers of the angular momenta. In this Letter we set $j \equiv j_1 = j_2 = 9$; i.e., the dimension N is $2j + 1 = 19$. As the value of k_i increases, the classical dynamics of the single top changes as follows : while the phase space is not fully chaotic and tori remain for $k_i \lesssim 3$, almost all classical trajectories are chaotic for $k_i \gtrsim 6$. In our study, we will vary k_i to see how the statistical properties of

TABLE I. The features of the classical trajectories starting from the initial conditions, $(\theta_a, \phi_a) = (0.89, 0.63)$ and $(\theta_b, \phi_b) = (2.25, 0.63)$. The characters c and t stand for chaos and torus, respectively. The centers of the quantum initial states, A and B, are located there.

	k_1	k_2	AA	AB	BA	BB
Figure 1	7.0	6.5	cc	cc	cc	cc
Figure 2	7.0	2.5	cc	ct	cc	ct
Figure 3	3.0	2.5	cc	ct	tc	tt

entanglement emerge, and choose $c = 0.35$ so that the phase space structure of the single kicked top is reflected in the coupled system.

In our quantum calculations, the initial states are taken to be the products of two spin-coherent states [21] A and B located at $(\theta_a, \phi_a) = (0.89, 0.63)$ and $(\theta_b, \phi_b) = (2.25, 0.63)$, respectively. The spin-coherent states reflect classical-quantum correspondence. For the classical single kicked top, while (θ_a, ϕ_a) is located within the chaotic sea for $k_i \gtrsim 2.5$, (θ_b, ϕ_b) is in a torus for $k_i \lesssim 3$ and in the chaotic sea for $k_i \gtrsim 6$. Thus, the dependence on the initial states also reveals how the difference of dynamics is reflected in entanglement. In the following, we abbreviate these four choices as AA, AB, BA, and BB, respectively. For each of the initial states, we calculate the time evolution of the wave function, and construct an ensemble of 10^5 wave functions over an interval from $t = 10^4$ to $t = 1.1 \times 10^5$. From the ensembles of the wave functions, we evaluate the one-level distribution of the Schmidt eigenvalues. In Figs. 1–3, we show the distributions for $(k_1, k_2) = (7.0, 6.5)$, (7.0, 2.5), and (3.0, 2.5), respectively. Correspondence between the classical and quantum initial conditions is summarized in Table I. In Figs. 1–3, we also plot $P^{(F)}_{1,N}$ and $P^{(L)}_{1,N}$ for comparison.

In Fig. 1, both of the tops are fully chaotic. For all the initial states, the centers are located in the classical chaotic sea. The distributions exhibit almost identical forms irrespective of the different initial conditions. Moreover, they agree well with $P^{(F)}_{1,N}$ given by Eq. (5) in the soft edge as well as the hard edge and the bulk region. Thus, the dynamically produced entanglement for the coupled system of fully chaotic tops has universality described by the random matrix theory of the fixed-trace ensemble. Note the difference between $P^{(F)}_{1,N}$ and $P^{(L)}_{1,N}$. The peaks of $P^{(F)}_{1,N}$ are more pronounced compared to those of $P^{(L)}_{1,N}$, although both $P^{(F)}_{1,N}$ and $P^{(L)}_{1,N}$ show an oscillatory behavior and the locations of their peaks are very near. This means that the level-repulsion in the fixed-trace ensemble is more enhanced than that in the Laguerre ensemble.

In Fig. 2, one of the tops is fully chaotic but the other not. All of the distributions deviate from that of the fixed-trace ensemble. For the initial states AA and BA, however, the distributions show similar forms and their deviations are smaller. Note that these initial states are the products of the states located in the chaotic sea. In Fig. 3, neither of the tops is fully chaotic. The dependence on the initial states becomes larger; while the distribution for AA is relatively similar to $P^{(F)}_{1,N}$, that for BB does not even show the oscillatory behavior. Note that the distributions for AB and BA have similar forms. This similarity is understandable because both of them are the products of one state in the chaotic sea and the other in the torus, and the kicks of the coupled tops have similar strengths. Thus, the degree of the deviations reflects the features of the classical dynamics.

In this Letter, we have obtained the analytic form Eq. (5) of the one-level distribution of the Schmidt eigenvalues for the fixed-trace ensemble. The derivation utilizes the theory

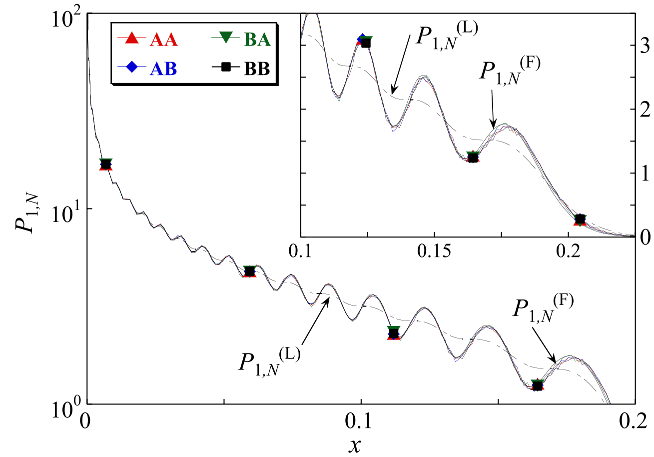

FIG. 1 (color online). The distributions of the Schmidt eigenvalues for the coupled kicked tops. The parameters are $j = 9$, $k_1 = 7.0$, $k_2 = 6.5$, and $c = 0.35$; i.e., both of the tops are fully chaotic. The four initial conditions are AA, AB, BA, or BB. For comparison, the distributions for the fixed-trace ensemble $P^{(F)}_{1,N}$ and the Laguerre ensemble $P^{(L)}_{1,N}$ are shown, respectively. The vertical axis is logarithmic and that in the inset is linear.

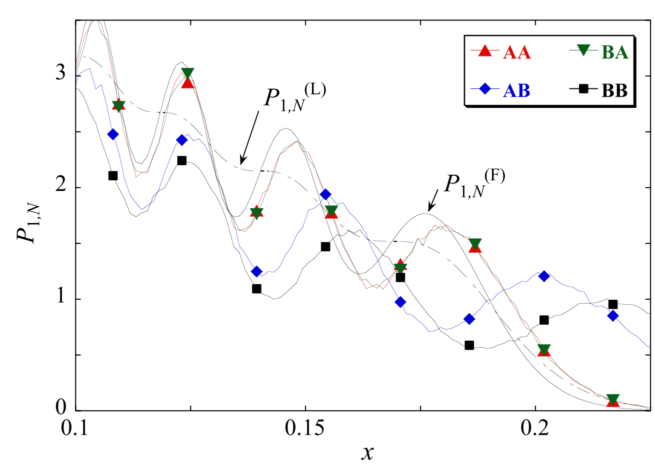

FIG. 2 (color online). The distributions of the Schmidt eigenvalues for the coupled kicked tops. Here, we set $k_1 = 7.0$, $k_2 = 2.5$; i.e., the first top is fully chaotic and the second is not. The rest of the parameters are the same as in Fig. 1.

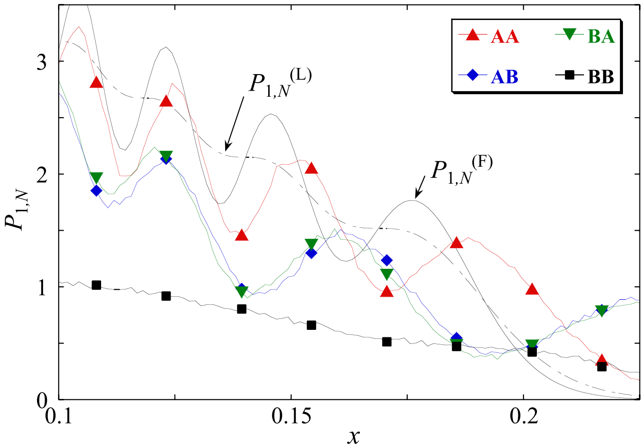

FIG. 3 (color online). The distributions of the Schmidt eigenvalues for the coupled kicked tops. Here, we set $k_1 = 3.0$, $k_2 = 2.5$; i.e., neither of the tops is fully chaotic. The rest of the parameters are the same as in Fig. 1.

of the holonomic system of differential equations developed in [18]. We have found that Eq. (5) agrees with the distribution of the Schmidt eigenvalues for a quantum system over the whole range when the system involves fully chaotic dynamics. It indicates that the fixed-trace ensemble describes the statistical properties of entanglement for quantum chaotic systems, thereby leading us to conclude that the statistical properties are universal [22].

The existence of the universality means that states of quantum chaos attain "equilibrium," which is insensitive to the details of coupled systems. In other words, the statistical properties of entanglement exhibit the same features as far as chaos is strong enough. Moreover, we suggest that the statistical properties of entanglement in equilibrium does not depend on how the total system is divided into two subsystems. Thus, we expect that these features of equilibrium are the origin of decoherence.

We will extend our analysis to the case when the dimensions of subsystems differ significantly [19]. Then, the asymptotic analysis will reveal the behavior of systems in heat bath. We will also study analytic forms of correlations of the Schmidt eigenvalues. They will be of importance not only in equilibrium but also in relaxation toward equilibrium. These studies are in progress and will be published elsewhere.

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Note added.—We have succeeded in reducing the polynomial $Q_n(x)$ given by Eqs. (6)–(10) to a simpler form.

*kubotani@yukawa.kyoto-u.ac.jp
†adachi@aa.ap.titech.ac.jp
‡toda@ki-rin.phys.nara-wu.ac.jp

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