Chaos-Assisted Tunneling and $1/f^{\alpha}$ Spectral Fluctuations in the Order-Chaos Transition

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It has been shown that the spectral fluctuations of different quantum systems are characterized by $1/f^{\alpha}$ noise, with $1 \leq \alpha \leq 2$, in the transition from integrability to chaos. This result is not well understood. We show that chaos-assisted tunneling gives rise to this power-law behavior. We develop a random matrix model for intermediate quantum systems, based on chaos-assisted tunneling, and we discuss under which conditions it displays $1/f^{\alpha}$ noise in the transition from integrability to chaos. We conclude that the variance of the elements that connect regular with chaotic states must decay with the difference of energy between them. We compare the characteristics of the transition modeled in this way with what is obtained for the Robnik billiard.

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The quantum transition from integrability to chaos gives rise to a change in the statistical properties of the energy levels. Integrable systems display uncorrelated adjacent energy levels, which are well described by Poisson statistics [1]. On the contrary, fully chaotic systems show strong repulsion between energy levels and follow the prediction of Random Matrix Theory (RMT) [2]. However, typical systems in nature are neither integrable, nor fully chaotic, but somewhere in between; they are partially chaotic or *mixed*. The spectral statistics of such kinds of systems is not well understood. In the semiclassical limit ($\hbar \to 0$), it is well described by the Principle of Uniform Semiclassical Condensation (PUSC) of Wigner function of eigenstates [3,4]; but this theory constitutes a good approximation only if the energy is sufficiently large. Tunneling effects between chaotic and regular states provide a more complete description [5] (see below for details).

One of the main features of mixed systems is the fractional power-law level repulsion (see [6] and references therein), which cannot be explained by PUSC. Level repulsion is an universal characteristic of chaotic systems; it implies that the distance between two adjacent energy levels, $s_i = \epsilon_{i+1} - \epsilon_i$ [7], cannot be equal to zero, and $P(s) \to s$ for $s \to 0$. In many mixed systems, $P(s) \to s^{\beta}$, with $0 \leq \beta \leq 1$, for $s \to 0$.

Recently, another fractional power-law behavior has been identified. The sequence of energy levels can be considered as a time series, where the energy plays the role of time. Following this analogy, it is well established that fully chaotic systems give rise to $1/f$ noise, whereas integrable systems are characterized by $1/f^2$ noise [9,10]. For the intermediate regime, it has been found that different quantum systems display $1/f^{\alpha}$ noise, with $1 \leq \alpha \leq 2$ [11,12]. (Note that the $1/f^{\alpha}$ behavior entails a similar fractional exponent in the form factor, $K(\tau) \propto \tau^{\beta}$, $\beta = 2 - \alpha$, for $\tau \ll 1$ [10].) This feature has not been previously explained; in particular, PUSC predicts a mixture of $1/f$ and $1/f^2$ behaviors [13]. Random matrix models give rise to a similar result [14,15].

Power-law level repulsion has been recently explained resorting to chaos-assisted tunneling. In a partially chaotic quantum system, different invariant tori can be connected by means of quantum dynamical tunneling, mediated by states in the chaotic part of the phase space [5]. This process can be modeled with random matrix ensembles, in which independent Gaussian random variables connect chaotic with integrable subspaces [5,16–18]. These models describe the level splitting distribution [16] and the power-law level repulsion [18,19] in mixed systems. Recently, a semiclassical formalism has been derived to calculate the variance of the connecting elements [17,20].

In this Letter, we show that chaos-assisted tunneling can also explain the $1/f^{\alpha}$ behavior found in mixed systems. Our point of departure is a random matrix ensemble in which all the elements are independent Gaussian random variables. The key issue of our model is that, contrary to what it is usual [5,16–21], the variance of the connecting elements is not a constant, but it depends on the difference of energy between regular and chaotic states. Our results suggest that it is mandatory to take into account this fact to obtain the $1/f^{\alpha}$ behavior characteristic of the transition from integrability to chaos.

The Hamiltonian of a mixed system can be written in the following form

$$H = \sum_R E_R |\Psi_R\rangle\langle\Psi_R| + \sum_R E_C |\Psi_C\rangle\langle\Psi_C| + \sum_{RC} \{V_{CR} |\Psi_C\rangle\langle\Psi_R| + c.c.\}, \quad (1)$$

where E_R and Ψ_R are the energies and wave functions of regular states; E_C and Ψ_C, the energies and wave functions of chaotic states; and V_{RC} describes the interaction between regular and chaotic states. A random matrix model for this Hamiltonian is

$$H = \begin{pmatrix} \text{GOE} & V \\ V & \text{GDE} \end{pmatrix}. \quad (2)$$

Gaussian Orthogonal Ensemble (GOE) and Gaussian Diagonal Ensemble (GDE) represent square diagonal submatrices, whose elements are the eigenvalues of a matrix belonging to the GOE and a matrix belonging to the GDE, respectively; they describe the chaotic and the regular energy levels. V determines the tunneling rate between these states.

If we set V to zero, this model implements the scenario of PUSC, giving rise to the Berry-Robnik nearest neighbor distribution [4]. Its only free parameter is the ratio between the number of chaotic N_C and regular states N_R, which is equivalent to the ratio of the corresponding parts of the classical phase space, $\rho = N_R/(N_C + N_R)$.

To model chaos-assisted tunneling, we introduce a V matrix composed by independent Gaussian random variables. The variance of these elements is usually taken as a constant, estimated by means of semiclassical arguments [17,19,20] or fitted as a free parameter [5,16,18,21]. Such a model reproduces the fractional level repulsion, but, as we will see below, it does not give rise to $1/f^\alpha$ noise in the transition form order to chaos. Here, we consider a more complex model: we assume that the amplitude of the connecting depends on the difference of energy between the connected states $\langle \Psi_C|H|\Psi_R\rangle = f(|E^C - E^R|)$.

To implement this assumption, the elements V_{ij} are generated as Gaussian random variables, with a nonconstant variance $\sigma_{ij} = f(|\overline{E_i^C} - \overline{E_j^R}|)$. We build the model in terms of the average values of the energy levels $\overline{E_i^C}$ and $\overline{E_j^R}$ to assure that it is well defined by a single probability distribution—the sequences of actual chaotic $\{E_i^C\}$ and regular $\{E_j^R\}$ energy levels are different for each matrix.

Let us consider two levels at energies E_C and $E_R = E_C + d$ that have a finite width and are described by Gaussians with variances σ_R and σ_C, respectively. As tunneling only takes place between states with the same energy [5], it is reasonable to consider that the tunneling amplitude is proportional to the overlap between these two Gaussians, which can be measured as the area below their intersection. In general, this gives rise to an involved expression, which consists of a sum of several error functions, depending on σ_C, σ_R, and d; its decay is faster or slower in function of the specific values of these parameters. As a simplified model, we propose an exponential function

$$\sigma_{ij} = \exp(-\lambda|\overline{E_i^C} - \overline{E_j^R}|), \quad (3)$$

which only depends on two independent parameters: the ratio between regular and chaotic states, ρ, and the decay rate of the V elements, λ. This function has the advantage that it is more tractable than the sum of several error functions, giving rise to a similar decay, depending on the value of λ. As a more realistic model, we also propose

$$\sigma_{ij} = 1 - \mathrm{Erf}(\gamma|\overline{E_i^C} - \overline{E_j^R}|). \quad (4)$$

If we consider that the connected regular and chaotic levels have the same width σ, then $1/\gamma = \sqrt{2}\sigma$; thus, both parameters of the model ρ and γ can be directly related to physical magnitudes. We will see that both choices for σ_{ij} give rise to similar results.

What follows is a numerical diagonalization of this set of random matrices. We have built the chaotic part of the Hamiltonian diagonalizing GOE matrices with dimension $N = 3000\rho$, and the regular part generating $N = 3000(1 - \rho)$ uncorrelated random Gaussian variables. Both sequences have been rescaled such that their mean density of levels is equal to a Gaussian with $\mu = 0$ and $\sigma = 1$; thus, all the variables of the model are Gaussian. V matrices have been built considering that, in this case, $\overline{E_i^{R,C}} = \sqrt{2}\{\mathrm{Erf}^{-1}[(2i - N^{R,C})/N^{R,C}]\}$, where Erf^{-1} is the inverse of the error function. To avoid spurious effects due to the unfolding procedure, we analyze only sequences of 1000 levels coming from the central part of each spectrum.

To study the spectral fluctuations, we use the statistic $\delta_n = \epsilon_{n+1} - \epsilon_1 - n$, where $\{\epsilon_i\}$ is the sequence of unfolded energy levels [7]. We are interested in its power spectrum

$$P_k^\delta = \left|\frac{1}{\sqrt{N}}\sum_{n=1}^{N}\delta_n \exp\left(\frac{-2\pi i k n}{N}\right)\right|^2. \quad (5)$$

In Fig. 1, we show the average power spectrum $\langle P_k^\delta\rangle$ calculated with Eq. (3), for six representative values of ρ and λ; it is obtained averaging over 20 different matrices for each value of ρ and λ. Solid lines represent the least-square fit to a power law $\langle P_k^\delta\rangle = A/k^\alpha$. The results of the fitting procedure are shown in the fourth column of Table I. We can see that all the cases depicted display a power-law behavior of $\langle P_k^\delta\rangle$ for different values of α, which cover the complete transition from integrability to chaos. The first points do not follow this power law because the unfolding procedure introduces spurious effects in them [22]. There is also a small discrepancy between the numerical results and the power-law fit in the high-frequency region; this is expected to occur for any value of α, since the $1/f^\alpha$ behavior is an approximation which is not accurate in the high-frequency region for both fully chaotic ($\alpha = 1$) and integrable ($\alpha = 2$) limits [10]. Similar results are obtained with Eq. (4).

To check that this model also displays fractional level repulsion, we have fitted the numerical results to the Brody formula for the nearest neighbor spacing distribution [8], $P(s) \sim s^\omega$, $s \ll 1$. The results are shown in the fifth column of Table I. As it has been previously obtained in some systems [11], we can see that the $P(s)$ distribution changes faster than $\langle P_k^\delta\rangle$ statistic. This fact can be understood as follows. The $P(s)$ distribution measures the distance between consecutive energy levels; it takes into account only very short-range correlations. On the contrary, $\langle P_k^\delta\rangle$ statistic measures both short- and long-range correlations; the

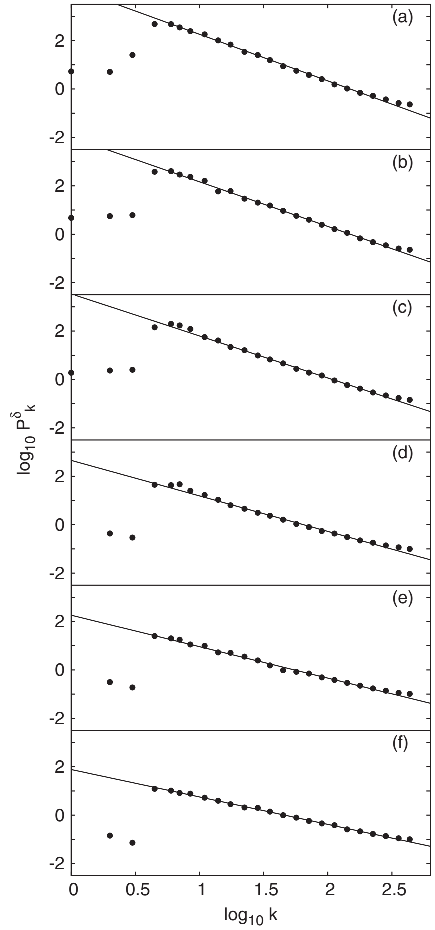

FIG. 1. $\langle P_k^\delta \rangle$ for six different values of ρ and λ (see Table I). Filled circles represent the average numerical results for 20 matrices of $N = 3000$; the solid line is the least-square fit to a power law $\langle P_k^\delta \rangle = A/k^\alpha$.

former ones determine the trend of the high-frequency region, and the last ones, the behavior of the low-frequency region. Thus, a global measure of all the frequencies, as the exponent α, cannot be univocally related to the Brody parameter.

In Fig. 2, we compare our model with one in which all the elements of V are generated with the same variance σ. We have chosen two intermediate cases between regularity and chaos: for our model, we use Eq. (3) and $\rho = 0.5$, $\lambda = 30$; for the ensemble characterized by a constant variance, $\rho = 0.5$, $\sigma = 2 \cdot 10^{-3}$. We plot the numerical results together with the power-law fit for the former ($\alpha = 1.58 \pm 0.01$), and the theoretical predictions for chaotic and regular spectra [10]. We can see that the ensemble with a constant variance does not display a power-law behavior. Instead, the high-frequency region follows the chaotic behavior, whereas the low-frequency region is closer to the regular one. This entails that the transition does not affect uniformly the whole spectrum, but short-range correlations transit to the behavior of chaotic spectra faster than long-range ones; this feature defines a *critical scale* in which the transition takes place. In some systems, this scale has a physical meaning; in [14], it has been used to estimate the Thouless energy in the Anderson transition. On the contrary, in our model, the transition is manifested over all the scales, and therefore it is not characterized by a critical scale.

To test the applicability of our model, we analyze how the exponent α changes with the fraction of regular classical trajectories ρ. This calculation has been previously done for the Robnik billiard, which transits from integrability to fully developed chaos [11]. To compare the curve $\alpha(\rho)$ of this system with the one obtained with our model, we fix the decay rates of the connecting elements λ and γ and change only ρ along the transition. This procedure gives rise to a plausible physical scenario; as we have no information about the real coupling between the regular and chaotic states in the Robnik billiard, it is reasonable to assume that it does not change with ρ. The results are shown in Fig. 3. We can see that the Robnik billiard and our model display the same qualitative behavior. In both cases, the major change in the exponent α happens for almost chaotic phase spaces (when the majority of the states are chaotic, in our model); for $\rho > 0.1$–0.2, all the curves change very slowly. From a quantitative point of view,

TABLE I. Results for the exponents α and ω for the cases plotted in Fig. 1.

Fig.	ρ	λ	α	ω
1a	0.8	5000	1.92 ± 0.03	0.053 ± 0.007
1b	0.6	1000	1.84 ± 0.03	0.056 ± 0.006
1c	0.3	250	1.74 ± 0.02	0.36 ± 0.04
1d	0.1	250	1.47 ± 0.03	0.89 ± 0.02
1e	0.1	100	1.30 ± 0.04	0.90 ± 0.02
1f	0.03	100	1.13 ± 0.02	0.94 ± 0.01

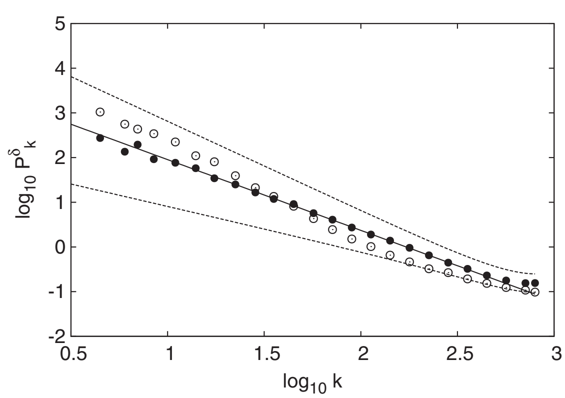

FIG. 2. Comparison between $\langle P_k^\delta \rangle$ obtained with Eq. (3), for $\lambda = 30$ (filled circles), and a similar model in which the variance of the elements of V is constant $\sigma = 2 \cdot 10^{-3}$ (empty circles); in both cases, $\rho = 0.5$. The solid line represents the power law $\langle P_k^\delta \rangle = A/k^\alpha$, $\alpha = 1.58 \pm 0.01$, and the dotted lines represent the theoretical predictions for GOE and GDE.

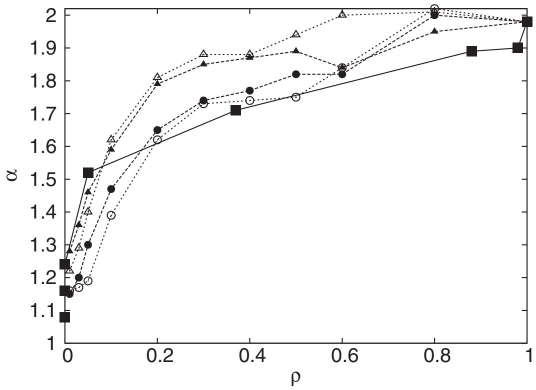

FIG. 3. Exponent α vs fraction of regular classical trajectories ρ, for the Robnik billiard [11] (squares), and several realizations of our model. Filled circles are obtained with Eq. (3), with $\lambda = 250$; for solid triangles, $\lambda = 1000$. Empty circles are obtained with Eq. (4), with $\gamma = 50$; for empty triangles, $\gamma = 200$.

the cases $\lambda = 1000$ and $\gamma = 200$ give a good description of the behavior of the Robnik billiard for $\rho < 0.1$; for larger values of ρ, the cases $\lambda = 250$ and $\gamma = 50$ give better results. The fact that both proposed decay rates, Eq. (3) and (4), give rise to the same qualitative behavior is a signature of the robustness of the model. The precise shape of the decay is needed for a quantitative fit, but the main features of the transition can be reproduced with several fastly decaying connecting elements.

In conclusion, we have shown that chaos-assisted tunneling can explain the $1/f^\alpha$ found in some quantum systems in the transition from integrability to chaos [11,12]. We have proposed a Gaussian random matrix ensemble which gives rise to this behavior. We have shown that the random elements which connect chaotic with regular states must not be generated with a constant variance, as it was previously done [5,16–21]. We have proposed instead a model in which the variance of these random elements decays fastly with the difference of energy between the connected states; this reflects the fact that chaos-assisted tunneling is only important when chaotic and regular states are close in energy [5]. With this model, we have obtained a complete transition from integrability to chaos, characterized by a power law $1/f^\alpha$ with $1 \le \alpha \le 2$. We have compared how the exponent α changes with the proportion of integrable states ρ in our model, with the previously published results for the Robnik billiard. We have concluded that our model reproduces the qualitative feature of the transition, for two different kinds of decay for the connecting elements. Moreover, a quantitative description can be obtained if we allow the rate of decay of the connecting elements λ or γ to change along the transition.

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