Combining Infrared and Low-Temperature Asymptotes in Yang-Mills Theories

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(Received 14 December 2007; published 3 June 2008)

We demonstrate that the powerlike nonperturbative behavior of gluon and ghost propagators in the infrared limit of Yang-Mills theories can provide at finite temperatures T a negative T^4 contribution to the pressure and energy density. The existence of a mass gap then implies new relations between the infrared critical exponents of gluon and ghost propagators.

DOI: 10.1103/PhysRevLett.100.222001 PACS numbers: 12.38.Aw, 11.15.Tk, 25.75.Nq

The zeros of Green's functions play an important role in various condensed matter systems ranging from superfluid ^{3}He to unconventional superconductors [1]. Under certain circumstances poles of Green's functions in momentum space, which determine the excitation spectrum of the system, can undergo transition to their opposites, that is, zeros. A known example of such a system is the Mott insulator. In this case, a pole of a one-particle Green's function becomes a zero as the system passes through the Mott transition [2]. In field theory, a similar example is provided by strong interactions. As first argued by Gribov [3], in the Landau gauge a perturbative pole of the gluon propagator is converted into a zero at the vanishing momentum [4,5]. This phenomenon is confirmed by both analytical calculations and numerical simulations [6]. The effect is related to the violation of reflection positivity and is a signature of the color confinement. The aim of this Letter is to demonstrate that anomalous behavior of the gluon propagator in the infrared region is directly related to the anomalous contribution of massless degrees of freedom at vanishing temperatures. The very existence of a mass gap in confining theories constitutes then a new kind of analyticity which constrains infrared asymptotes of the propagators.

We consider pressure P and energy density $\epsilon \equiv E/V$ of strongly interacting Yang-Mills fields at temperature T and in volume V. In the limit of vanishing coupling, or in the tree approximation,

$$\varepsilon_{\rm free}(T) = 3P_{\rm free}(T) = N_{\rm d.o.f.} C_{\rm SB} T^4, \qquad (1)$$

where $C_{\rm SB} = \pi^2/30$ is the Stefan-Boltzmann factor. There are $N_{\rm d.o.f.} = 2(N_c^2 - 1)$ degrees of freedom corresponding to $N_c^2 - 1$ gluons with two transverse polarizations.

Nonperturbative strong interactions are manifested in suppression of the energy density and pressure at $T \ll T_c$, where T_c is the temperature of the confinement-deconfinement phase transition:

$$\frac{\varepsilon(T)}{T^4} \ll 1, \qquad \frac{P(T)}{T^4} \ll 1 \quad \text{for } T \ll T_c. \qquad (2)$$

In particular, such a suppression is established via lattice simulations both in the pure gluonic $SU(3)$ case [7] and in a more realistic case which incorporates light quarks as well [8]. Hereafter we will concentrate on the pure gluonic $SU(3)$ gauge system.

The suppression (2) is a consequence of two intimately related properties of QCD, namely, confinement of color and mass-gap generation. Massless gluons are confined into glueballs which are massive. As a result, all the thermodynamic quantities are to be suppressed at low temperatures as $O(\exp\{-M/T\})$, where M is the mass of the lightest glueball; for a related discussion see [9].

The simplest quantity for a homogeneous system in the thermodynamic equilibrium is pressure,

$$P = \frac{T}{V} \log Z = P_{\rm free} + P_{\rm int}, \qquad (3)$$

where Z is the partition function and we explicitly separate the tree level contribution $P_{\rm free}$ given by Eq. (1). The second term $P_{\rm int}$ is a correction due to the interactions. The interaction part of the thermodynamic quantities can be written as a loop expansion, in terms of the full propagators and vertices; see, e.g., Ref. [10].

In order to make explicit calculations one generically needs a gauge fixing, and below we usually refer to the Landau gauge which—being both Lorentz and color-symmetric—is one of the best studied gauges nowadays. However, our approach is not limited to the Landau gauge, and, as we will see below, is in fact very general. As a consequence, the final result—a relation between the infrared exponents of the dressed gluon propagator and the dressed ghost propagator—should be gauge invariant, provided that the infrared exponents (which may vary from gauge to gauge) are defined as we discuss below.

Before going into details we would like to make a general comment about the gauge invariance. Despite the fact that Green functions in non-Abelian theories are essentially gauge-variant, their properties may be directly linked to the confinement. The best known example of this kind is provided by Ref. [3], which relates the linear confining potential increase of the color charge interaction at large distances to cancellation of the infrared pole of the gluon Green's function.

At finite temperatures, the gluon propagator $D^{ab}_{\mu\nu}$ and the ghost propagator G^{ab} in the Landau gauge are parametrized by three functions D_T, D_L, and D_G:

$$D^{ab}_{\mu\nu}(\mathbf{p}, p_4) = \delta^{ab}[P^T_{\mu\nu}D_T(\mathbf{p}, p_4) + P^L_{\mu\nu}D_L(\mathbf{p}, p_4)], \quad (4)$$

$$G^{ab}(\mathbf{p}, p_4) = -\delta^{ab}D_G(\mathbf{p}, p_4), \quad (5)$$

where $P^T_{\mu\nu}$ and $P^L_{\mu\nu}$ are projectors onto spatially transverse and spatially longitudinal parts [11].

A conspicuous feature of the propagators is the anomalous dressing of the gluon and ghost propagators in the infrared region [6]. In the low-temperature limit

$$D_i(p^2) = (p^2)^{\gamma_i - 1}H_i(p^2) \quad \text{as } p^2 \to 0, \quad (6)$$

where γ_i are infrared exponents and the functions H_i are finite at $p^2 \to 0$ in all the cases, $i = T, L, G$.

The infrared behavior of the propagators is best studied at zero temperature. First of all, the gluon infrared exponents are degenerate, $\gamma_D = \gamma_T \equiv \gamma_L$. We have already mentioned Gribov's scenario for $\gamma_D > 0$ [3–5]. Moreover, according to the Kugo-Ojima criterion of confinement [12] the ghost propagator is enhanced in the infrared, compared to its perturbative value:

$$\gamma_D + 2\gamma_G = 0. \quad (7)$$

There is some support for the validity of this relation, both from the analytical studies of the Dyson-Schwinger equations [6,13] and from numerical, lattice data [14,15]. However, a final conclusion on the validity of Eq. (7) seems to not have yet been reached; see in particular Refs. [13,16–18]. A cautionary remark comes also from a confining Abelian gauge model where one can prove analytically [19] that Eq. (7) does not hold.

Turning to nonzero temperatures, let us note that the infrared properties of the gluon propagators have already been used to study thermodynamics of Yang-Mills plasma at $T > T_c$ in the Coulomb gauge [20]. The idea is that constraining the configuration space through the gauge fixing condition leads to a change in the dispersion relations for gluons, or position of the poles of the propagator. In the Landau gauge, a relation between the nonperturbative Green's functions and the thermodynamic potential was discussed in Refs. [21,22]. Below we concentrate on zeros rather than poles of the propagators and demonstrate that zeros give rise to new T^4 terms in the equation of state.

To illustrate the basic idea consider first a toy model describing 1 degree of freedom, or a real scalar field $\phi(x)$ with a quadratic action:

$$S[\phi] = -\frac{1}{2}\int_0^{1/T} d\tau \int d^3x\phi(x)\mathcal{D}^{(-1)}(x-y)\phi(y), \quad (8)$$

where the propagator $\mathcal{D}(x - y)$ is defined as

$$\mathcal{D}(x - y) = \langle\phi(x)\phi(y)\rangle, \quad (9)$$

and we use the imaginary time formalism. The partition function can be calculated in the standard way [11]:

$$\mathcal{Z} = \int D\phi e^{-S^{(2)}[\phi]} = \exp\left\{\frac{1}{2}\,\mathrm{Tr}\log(T^2\mathcal{D})\right\}, \quad (10)$$

where we omit the irrelevant prefactor. Taking the trace over all the states, one gets for the pressure (3)

$$P = -\frac{T}{2}\int\frac{d^3p}{(2\pi)^3}\sum_{n\in\mathbb{Z}}\log[T^2\mathcal{D}(\mathbf{p}, p_4)]|_{p_4=\omega_n}, \quad (11)$$

where the sum runs over the Matsubara frequencies, $\omega_n = 2\pi nT$. The energy density is given by

$$\varepsilon = -T\int\frac{d^3p}{(2\pi)^3}\sum_{n\in\mathbb{Z}}\frac{\partial\log[p_4^2\mathcal{D}(\mathbf{p}, p_4)]}{\partial\log p_4^2}\bigg|_{p_4=\omega_n}. \quad (12)$$

The temperature dependent part of this sum can be evaluated following Refs. [11,23]:

$$\varepsilon(T) = -\frac{1}{\pi i}\int\frac{d^3p}{(2\pi)^3}\int_{-i\infty+\epsilon}^{+i\infty+\epsilon}dp_0 f_T(p_0) \times\frac{F(p_0) + F(-p_0)}{2} \quad (13)$$

$$F_\varepsilon(p_0) = \frac{\partial\log[p_4^2\mathcal{D}(\mathbf{p}, p_4)]}{\partial\log p_4^2}\bigg|_{p_4=-ip_0}, \quad (14)$$

where $f_T(p_0) = 1/(e^{p_0/T} - 1)$ is the Bose-Einstein distribution for a particle with energy $\omega = p_0$. Equation (13) is valid provided the analytical function $F(p_0)$ does not have poles at purely imaginary p_0. Since the integrand in Eq. (13) is fast converging as $p_0 \to +\infty$, the contour of integration can be closed as a semicircle in the right half of the p_0-complex plane, reducing the integral (13) to contribution of poles of the function $F_\varepsilon(p_0)$.

Poles of the propagator $\mathcal{D}$ become poles of the function $F_\varepsilon(p_0)$ and contribute to $\epsilon(T)$. This agrees with our intuition since the energy spectrum of the model (8) is determined by the poles of the propagator $\mathcal{D}$. However, the crucial point is that not only the poles of the propagator but also its *zeros* contribute to the energy density (13) and (14). In fact, zeros of the propagator $\mathcal{D}$ become poles of the function $F_\varepsilon(p_0)$ as well.

Imagine that the propagator (9) in the infrared region has the same criticality as the gluon or ghost propagators (6) in Yang-Mills theory:

$$D^{(\gamma)} = \mathrm{const}(p^2)^\gamma/(p^2 + m^2). \quad (15)$$

Then we get from Eq. (14) the expression

$$F_\varepsilon^{(\gamma)}(p_0) = \frac{\omega_\mathbf{p}^2}{\omega_\mathbf{p}^2 - p_0^2} - \gamma\frac{p_0^2}{\mathbf{p}^2 - p_0^2}, \quad (16)$$

with two poles at the $\mathrm{Re}p_0 > 0$ half of the complex p_0

plane, at $p_0 = \omega_{\mathbf{p}}$ and at $p_0 = |\mathbf{p}|$. The residues, respectively, are $-\omega_{\mathbf{p}}/2$ and $\gamma|\mathbf{p}|/2$. The energy density (12) is

$$\varepsilon(T) = \varepsilon^{\text{free}}(T, m) - \gamma \varepsilon^{\text{free}}(T, 0), \tag{17}$$

$$\varepsilon^{\text{free}}(T, m) = \int \frac{d^3 p}{(2\pi)^3} \omega_{\mathbf{p}} f_T(\omega_{\mathbf{p}}). \tag{18}$$

Here $\varepsilon^{\text{free}}$ is the energy density of free relativistic particles of mass m.

It is clear from Eq. (17) that the infrared suppression (with $\gamma > 0$) in the particle propagator acts oppositely to the pole: the suppression "subtracts" *massless* degrees of freedom (with the dispersion relation corresponding to a relativistic massless "particle" $\omega_{\mathbf{p}} = |\mathbf{p}|$ and with $\varepsilon_{\text{free}}(T, 0) \propto T^4$) while a pole "supplies" them to the spectrum. The same effect is also from the expression for the pressure (11) because $\log D^{(\gamma)} = \gamma \log(p^2) - \log(p^2 + m^2)$, up to a momentum-independent term. Note that a zero of the propagator corresponds to a *negative* pressure and energy density. Thus, vanishing of the propagator in the infrared with critical exponent γ acts as a "reservoir" for γ massless degrees of freedom where the quantity γ is, in general, noninteger. As we see from our illustrative example, the low-temperature asymptotics of the thermodynamic functions are determined by the infrared behavior of the particle propagator.

Let us now turn back to Yang-Mills theory. In the leading, one-loop approximation (in terms of dressed propagators) pressure in Yang-Mills theory at vanishingly low temperatures is given by expression (11) with

$$\mathcal{D}(p^2) \to \left(\frac{D_L(p^2) D_T^2(p^2)[D_L(p^2) + D_T(p^2)]}{D_G^2(p^2)}\right)^{N_c^2 - 1}. \tag{19}$$

The ghost structure function D_G in the denominator of Eq. (19) corresponds to the Faddeev-Popov determinant which subtracts unphysical degrees of freedom. Indeed, if one neglects the interactions, $g = 0$, then

$$D_L^{\text{free}} = D_T^{\text{free}} = D_G^{\text{free}} = 1/p^2, \tag{20}$$

and therefore the energy and pressure are given by Eq. (1) with $N_{\text{d.o.f.}} = 2(N_c^2 - 1)$, as expected.

However, if one takes into account the infrared dressing of the propagators (6), then one gets for the pressure

$$P = (2 - \gamma_L - 2\gamma_T + 2\gamma_G - \min[\gamma_L, \gamma_T]) \times (N_c^2 - 1) C_{\text{SB}} T^4 + \cdots, \tag{21}$$

where ellipsis denote subleading $O(T^\kappa)$ terms with $\kappa > 4$, associated with cuts in the complex p_0 plane. The suppression (2) of thermodynamic quantities at low temperatures implies a relation between the infrared exponents:

$$\gamma_L + 2\gamma_T - 2\gamma_G + \min[\gamma_L, \gamma_T] - 2 = 0. \tag{22}$$

Since the functions H_i, $i = L, T, G$, are regular at $p^2 = 0$ the poles and/or zeros of these functions do not contribute to a T^4 term at very low temperatures. Indeed, poles and/or zeros at momenta $\text{Re}\, p_0 > 0$ would correspond to massive-like contributions which are exponentially suppressed by the Bose-Einstein factor as $T \to 0$.

Coming back to the question of gauge dependence of our results, we mention that the form of the propagators (6) assumes local Lorentz isotropy of the gauge condition at zero temperature. However, in a general case (like, for example, in the Coulomb gauge) and/or at a finite temperature (as we discuss in detail below) the form of the propagator is not rotationally invariant, and the rôle of the infrared critical exponent is played by the degree γ of the zero, corresponding to the free dispersion relation $p_0^{\text{free}} = |\mathbf{p}|$. In the Coulomb gauge effects of the fundamental modular region lead to a substantial modification of the gluon dispersion relation [20]. Namely, the relation between the energy and momentum of the gluon is no longer linear at low momenta. Thus, the gluon's contribution to the equation of state in this gauge does not contain T^4 terms in the limit of low temperature, in agreement with lattice simulations [20].

The relation (22) provides a new constraint on the critical exponents. Existing lattice data do not allow one to check it directly. One can try to unify (21) with other relations suggested in the literature. Assuming validity of (7) and $\gamma_L = \gamma_T$ we get a system of constraints which allows one to uniquely fix all the exponents and get what we would call a simplified solution, $\gamma_D = 2/5$ and $\gamma_G = -1/5$.

The positivity of γ_D, exhibited by this solution, is consistent with the infrared suppression of the gluon propagator expected on other grounds [6]. Also, the ghost exponent is consistent with the numerical result of Ref. [17]. However, for the simplified solution the gluon infrared exponent has too small of a value to guarantee vanishing of the gluon propagator at $p = 0$. Also, the simplified solution is in variance with results of Refs. [13,14]. One of the reasons could be violation at the presently available lattices of the equality $\gamma_L = \gamma_T$ which is granted at zero temperature. Indeed, according to numerical results of Ref. [22] the transverse-longitudinal degeneracy is not reached yet even at the lowest nonzero temperatures available now. One might guess that the discrepancy between the simplified solution and the existing simulations is rooted in the fact that the lowest currently available nonzero temperatures are still too high to be compared with our analytical predictions.

One can ask a question whether the relation (22) between the critical exponents is affected by the presence of quarks. As far as the quarks are massive, both numerical [24] and analytical results [25] indicate absence of the infrared particularities (zeros or poles) in the quark structure functions in the Landau gauge. Consequently our relation (22) is insensitive to the presence of the massive quarks. This conclusion agrees nicely with the result of Ref. [25]. However, for *exactly* massless quarks the T^4 terms do appear in the thermodynamical quantities in the $T \to 0$ limit because of the presence of *exactly* massless Nambu-Goldstone particles in the spectrum. Then the right-hand side Eq. (22) should be equal to $-N_{NG}/(N_c^2 - 1)$, where N_{NG} is the number of the Nambu-Goldstone bosons.

To summarize, the zeros in the momentum-space Green function are as important for the thermodynamics as the poles. Although zeros do not correspond to any states in the spectrum they contribute to the equation of state. The contribution of zeros in the gluon propagator to coefficients in front of T^4 terms in the energy density and pressure is negative. This observation can be interpreted in such a way that zeros correspond to confinement, or binding of the originally massless gluons into massive glueballs. Zeros of the ghost propagator, on the contrary, effectively supply massless degrees of freedom into the equation of state. Moreover, the very existence of the mass gap in confining theories requires vanishing of the overall contribution of massless degrees of freedom from the low-temperature equation of state. Hence, there arises the constraint (22) on the infrared exponents. Confinement resolves itself into a new kind of analyticity.

We are grateful to A. Di Giacomo for reading the manuscript and for comments, to E.-M. Ilgenfritz, A. Maas, and T. Suzuki for communications, and to G. E. Volovik for bringing to our attention Refs. [1,2]. We are thankful to the members of the Institute for Theoretical Physics of Kanazawa University and Research Institute for Information Science and Education of Hiroshima University for hospitality and stimulating environment. This work was partly supported by Federal Program of the Russian Ministry of Industry, Science and Technology No. 40.052.1.1.1112. M. N. C. is supported by the JSPS Grant No. L-06514, by Grant-in-Aid for Scientific Research by Monbu-kagakusyo, No. 13135216, and by a CNRS grant.

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