Exact Linear Hydrodynamics from the Boltzmann Equation

I. V. Karlin,[2,*] M. Colangeli,[1] and M. Kröger[1,†]

[1]*Polymer Physics, Department of Materials and Materials Research Center, ETH Zürich, CH-8093 Zürich, Switzerland*
[2]*Aerothermochemistry and Combustion Systems Lab, ETH Zürich, CH-8092 Zürich, Switzerland*
and School of Engineering Sciences, University of Southampton, SO17 1BJ, Southampton, United Kingdom
(Received 17 January 2008; published 29 May 2008)

Exact (to all orders in Knudsen number) equations of linear hydrodynamics are derived from the Boltzmann kinetic equation with the Bhatnagar-Gross-Krook collision integral. The exact hydrodynamic equations are cast in a form which allows us to immediately prove their hyperbolicity, stability, and existence of an *H* theorem.

Hydrodynamics assumes that a state of a fluid is solely described by five fields: density, momentum and temperature. Derivation of the Navier-Stokes-Fourier (NSF) hydrodynamic equations from the Boltzmann kinetic equation as the first-order approximation in the Knudsen number (ratio between mean free path and a flow scale) by Enskog and Chapman is a textbook example of a success of statistical physics [1]. Recent renewed interest to the problems beyond the standard hydrodynamics is due, in particular, to flow simulation and experiments at a micro- and nanoscale [2–4]. However, almost a century of effort to extend the hydrodynamic description beyond the NSF approximation failed even in the case of small deviations around the equilibrium. In order to appreciate the problem, let us remember that, in the NSF approximations, the decay rate of the hydrodynamic modes is quadratic in the wave vector, $\text{Re}(\omega) \sim -k^2$, and is unbounded. On the other hand, Boltzmann's collision term features equilibration with finite characteristic rates. This "finite collision frequency" is obviously incompatible with the arbitrary decay rates in the NSF approximation: intuitively, hydrodynamic modes at large k cannot relax faster than the collision frequency. Now, the classical method of Enskog and Chapman extends the hydrodynamics beyond the NSF in such a way that the decay rate of the next order approximations (Burnett and super-Burnett) are polynomials of higher order in k. In such an extension, relaxation rate may become completely unphysical (amplification instead of attenuation), as first shown by Bobylev [5] for a particular case of Maxwell molecules. This indicates the inability of the Chapman-Enskog method to tackle the above problem, and nonperturbative approaches are sought. The problem of exact hydrodynamics has been studied in depth recently for toy (finite-dimensional) models—moment systems of Grad—in [6,7], and many remarkable results were obtained. In particular, in [7] it was shown that the exact hydrodynamic equations are hyperbolic and stable for all wave numbers. However, for "true" kinetic equations such questions remain open.

In this Letter we derive exact hydrodynamic equations from the linearized Boltzmann equation with the Bhatnagar-Gross-Krook (BGK) collision term. This kinetic equation remains popular in applications [8], and features a single relaxation rate. The result for the hydrodynamic modes is demonstrated in Fig. 1. It is clear from Fig. 1 that the relaxation of none of the hydrodynamic modes is faster than $\omega = -1$ which is the collision frequency in the units adopted in this paper. Thus, the result for the exact hydrodynamics indeed corresponds to the above intuitive picture. Below, we apply the method of invariant manifold [9] to derive the hydrodynamic equations. The nonperturbative derivation is made possible with an optimal combination of analytical and numerical approaches to solve the invariance equation.

Point of departure is the linearized Boltzmann-BGK equation for the deviation $\triangle f = f - f^{\text{GM}}$ of the distribution function f from a global Maxwellian $f^{\text{GM}}(c^2) = \pi^{-3/2} e^{-c^2}$. In the reciprocal space, it reads,

$$\partial_t \Delta f = -i\mathbf{k} \cdot \mathbf{c} \Delta f - \delta f; \qquad \delta f = f - f^{\text{LM}}, \quad (1)$$

with the wave vector $\mathbf{k} = k\mathbf{e}_\parallel$ defining $\mathbf{e}_\parallel$, $k \equiv |\mathbf{k}|$, peculiar velocity $\mathbf{c}$ and time t. All quantities are considered dimensionless, i.e., reduced with the units of the relaxation
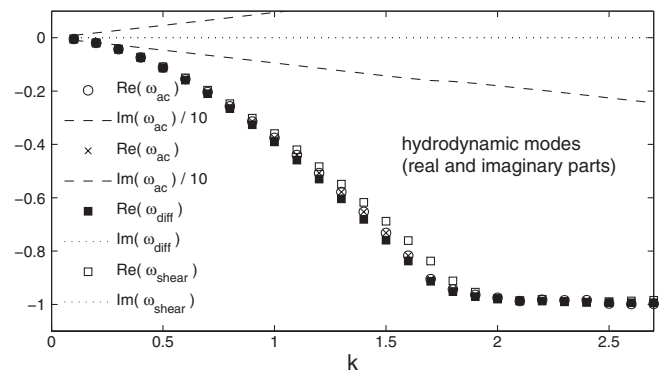

FIG. 1. Exact hydrodynamic modes ω of the Boltzmann-BGK kinetic equation as a function of wave number k (two complex-conjugated acoustic modes ω_{ac}, twice degenerated shear mode ω_{shear} and thermal diffusion mode ω_{diff}). The nonpositive decay rates $\text{Re}(\omega)$ attain the limit of collision frequency (−1) as $k \to \infty$.

time τ, the thermal velocity $\sqrt{2k_B T/m}$ and mass m of the particle. In (1), the linearized local Maxwellian is $f^{\mathrm{LM}} = f^{\mathrm{GM}}(1 + \varphi_0)$ where $1 + \varphi_0 = \langle 1 \rangle_f + 2\mathbf{c} \cdot \langle \mathbf{c} \rangle_f + \frac{2}{3} \times (c^2 - \frac{3}{2})\langle c^2 - \frac{3}{2} \rangle_f$. Averages are defined for arbitrary $\boldsymbol{\omega}$ via $\langle \boldsymbol{\omega} \rangle_f \equiv \int \boldsymbol{\omega} f d^3 c$, and we introduce pertinent quantities which characterize deviation from the global equilibrium: $n \equiv \langle 1 \rangle_f - 1$ (density perturbation), $\mathbf{u} \equiv \langle \mathbf{c} \rangle_f$ (velocity perturbation) and $T \equiv \frac{2}{3}\langle c^2 - \frac{3}{2} \rangle_f$ (temperature perturbation). Since the scalar product between $\mathbf{k}$ and $\mathbf{c}$ appears in (1), the distribution function offers symmetry with respect to the $\mathbf{e}_\parallel$-axis, which is not uniaxial in case $\mathbf{u}$ is not collinear with $\mathbf{e}_\parallel$. We denote the two components of the mean velocity as $u^\parallel = \mathbf{u} \cdot \mathbf{e}_\parallel$ and $u^\perp = \mathbf{u} \cdot \mathbf{e}_\perp$, where the unit vector $\mathbf{e}_\perp$ belongs to the intersection of the plane perpendicular to $\mathbf{k}$ and the plane spanned by $\mathbf{k}$ and $\mathbf{u}$, so that $\mathbf{u} = u^\parallel \mathbf{e}_\parallel + u^\perp \mathbf{e}_\perp$. Equations of change for moments $\langle \boldsymbol{\omega} \rangle$ are obtained by integration of the weighted (1) over $d^3 c$ as

$$\partial_t \langle \boldsymbol{\omega} \rangle_{\Delta f} = -i\mathbf{k} \cdot \langle \mathbf{c}\boldsymbol{\omega} \rangle_{\Delta f} - \langle \boldsymbol{\omega} \rangle_{\delta f}. \quad (2)$$

In order to calculate such averages, we can switch to spherical coordinates. For each (at present arbitrary) wave vector, we choose the coordinate system in such a way that its z direction aligns with $\mathbf{e}_\parallel$. We can then express $\mathbf{c}$ in terms of its norm c, a vertical variable z and plane vector $\mathbf{e}_\phi$ (azimuthal angle $\mathbf{e}_\phi \cdot \mathbf{e}_\perp = \cos\phi$) for the present purpose as $\mathbf{c}/c = \sqrt{1 - z^2}\mathbf{e}_\phi + z\mathbf{e}_\parallel$. We could have equally chosen a fixed coordinate system in the plane orthogonal to $\mathbf{k}$, and two fields instead of $u^\perp$ plus an angle, viz. $u_x^\perp + iu_y^\perp = e^{i\phi}u^\perp$. Because of isotropy, $u^\perp$ alone fully represents the twice degenerated (shear) dynamics. In order to simplify notation and compute the dynamics of all five fields we introduce a four-dimensional vector of hydrodynamic fields, $\mathbf{x} \equiv (x_1, x_2, x_3, x_4) = (\mathbf{x}^\parallel, x_4)$ with $\mathbf{x}^\parallel \equiv (n, u^\parallel, T)$ and $x_4 = u^\perp$. Then φ_0 takes a simple form, $\varphi_0 = \mathbf{X}^0 \cdot \mathbf{x}$. The vector $\mathbf{X}^0$ can immediately be read off, we have $\mathbf{X}^0(c, z) = (1, 2c_\parallel, c^2 - \frac{3}{2}, 2c_\phi)$, where we introduced, for later use, the abbreviations

$$c_\parallel \equiv \mathbf{c} \cdot \mathbf{e}_\parallel, \qquad c_\phi \equiv \mathbf{c} \cdot \mathbf{e}_\perp, \qquad c_\perp \equiv \frac{c_\phi}{\mathbf{e}_\perp \cdot \mathbf{e}_\phi}, \quad (3)$$

such that $i\mathbf{k} \cdot \mathbf{c} = ikc_\parallel$, $\mathbf{c} = c_\perp \mathbf{e}_\phi + c_\parallel \mathbf{e}_\parallel$ with $c_\parallel = cz$ and $c_\perp = c\sqrt{1 - z^2}$, contrasted by c_ϕ (and $\mathbf{e}_\phi$), do not depend on the azimuthal angle. Similarly, we introduce yet unknown fields $\delta\mathbf{X}(\mathbf{c}, \mathbf{k})$ which characterize the nonequilibrium part of the distribution function, $\delta\varphi = \delta f/f^{\mathrm{GM}}$ in terms of the hydrodynamic fields $\mathbf{x}$ themselves,

$$\delta\varphi = \delta\mathbf{X} \cdot \mathbf{x} = \delta X_1 n + \delta X_2 u^\parallel + \delta X_3 T + \delta X_4 u^\perp, \quad (4)$$

where δX_4 factorizes as $\delta X_4(c, z, \phi) = 2\delta Y_4(c, z)\mathbf{e}_\phi \cdot \mathbf{e}_\perp$. This "eigenclosure" (4) which formally and very generally addresses the fact, that we wish to *not* include other than hydrodynamic variables implies a closure between moments of the distribution function, to be worked out in

detail below. It assumes the existence of an invariant manifold, and the hydrodynamic fields as slow variables which leave the higher moments "slaved." It is an important technical aspect of our derivation to work with an orthogonal set of basis functions (irreducible tensors) to represent δf uniquely. Accordingly, $\langle \mathbf{X}^0 \rangle_{\delta f} = 0$ holds and the lowest order moments $\mathbf{x}$ are defined through the local Maxwellian part of the distribution function. Using the above form for δf in (1), and using the canonical abbreviation $\triangle \mathbf{X} \equiv \mathbf{X}^0 + \delta\mathbf{X}$, yields

$$\Delta\mathbf{X} \cdot \partial_t \mathbf{x} = -ikc_\parallel \Delta\mathbf{X} \cdot \mathbf{x} - \delta\mathbf{X} \cdot \mathbf{x}, \quad (5)$$

which is a nonlinear integral equation for the unknown fields $\delta\mathbf{X}$, because $\partial_t \mathbf{x}$ has to be replaced by the right hand side of (2), for a suitably chosen vector $\boldsymbol{\omega}$ fulfilling $\langle \boldsymbol{\omega} \rangle_{\Delta f} = \mathbf{x}$. Here, $\boldsymbol{\omega}$ is similar with $\mathbf{X}^0$ and differs from $\mathbf{X}^0$ mainly because of conventions for prefactors in the temperature and velocity definitions, $\boldsymbol{\omega} = (1, c_\parallel, \frac{2}{3} \times (c^2 - \frac{3}{2}), c_\phi)$. Within the same eigenclosure, Eq. (2) is linear in $\mathbf{x}$ and hence written as

$$\partial_t \mathbf{x} = \mathbf{M} \cdot \mathbf{x}. \quad (6)$$

The matrix $\mathbf{M}$ solely depends on the nonhydrodynamic fields, the heat flux $\mathbf{q} \equiv \langle \mathbf{c}(c^2 - \frac{5}{2}) \rangle_f$ and the stress tensor $\overline{\mathbf{cc}}$, where $\overline{\mathbf{s}}$ denotes the symmetric traceless part of a tensor $\mathbf{s}$ [7,10], $\overline{\mathbf{s}} = \frac{1}{2}(\mathbf{s} + \mathbf{s}^T) - \frac{1}{3}\mathrm{Tr}(\mathbf{s})\mathbf{I}$. Using (4), the stress tensor and heat flux uniquely decompose as follow

$$\boldsymbol{\sigma} = \sigma^\parallel \frac{3}{2}\overline{\mathbf{e}_\parallel \mathbf{e}_\parallel} + \sigma^\perp 2\overline{\mathbf{e}_\parallel \mathbf{e}_\perp}, \quad \mathbf{q} = q^\parallel \mathbf{e}_\parallel + q^\perp \mathbf{e}_\perp, \quad (7)$$

with the moments $\sigma^\parallel = (\sigma_1^\parallel, \sigma_2^\parallel, \sigma_3^\parallel) \cdot \mathbf{x}^\parallel$ and $\sigma^\perp = \sigma_4 x_4$, and similarly for $\mathbf{q}$ (see Row 2 of Table I). The prefactors arise from the identities $\overline{\mathbf{e}_\parallel \mathbf{e}_\parallel}$: $\mathbf{e}_\parallel \mathbf{e}_\parallel = \frac{2}{3}$ and $\overline{\mathbf{e}_\parallel \mathbf{e}_\parallel}$: $\mathbf{e}_\parallel \mathbf{e}_\perp = \frac{1}{2}$. The appearance of δY_4 rather than δX_4 in the expression for the orthogonal moment (in Table I) reflects the fact that we have already integrated out the angular variable, $\int_0^{2\pi} \mathbf{e}_\phi \mathbf{e}_\phi \cdot \mathbf{e}_\perp d\phi = \pi\mathbf{e}_\perp$. We note in passing that, while the stress tensor has, in general, three different eigenvalues, in the present symmetry adapted coordinate system it exhibits a vanishing first normal stress difference. Since the integral kernels of all moments in (7) do not depend on the azimuthal angle, these are actually two-dimensional integrals over $c \in [0, \infty]$ and $z \in [-1, 1]$, weighted by $2\pi c^2 f^{\mathrm{GM}}(c^2)\delta\mathbf{X}_\mu$.

Stress tensor and heat flux can yet be written in an alternative form which is defined by row 3 of Table I. As we will see later on, due to basic symmetry considerations, the hereby introduced functions A–Z are real valued. We postpone the related proof, and proceed by using these functions A–Z to split $\mathbf{M}$ into parts as $\mathbf{M} = \mathrm{Re}(\mathbf{M}) - i\mathrm{Im}(\mathbf{M})$,

$$\mathbf{M} = k^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ \frac{2}{3}X & 0 & \frac{2}{3}Y & 0 \\ 0 & 0 & 0 & D \end{pmatrix} - ik \begin{pmatrix} 0 & 1 & 0 & 0 \\ \tilde{B} & 0 & \tilde{C} & 0 \\ 0 & \tilde{Z} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (8)$$

TABLE I. Symmetry adapted components of (nonequilibrium) stress tensor $\boldsymbol{\sigma}$ and heat flux $\mathbf{q}$, both introduced in (7). Row 2: Microscopic expression of these components (averaging with the global Maxwellian). Short-hand notation used: $\lambda^{\|} = c_{\|}^2 - \frac{c^2}{3}$ and $\gamma^{\|} = (c^2 - \frac{5}{2})c_{\|}$. Row 3: Expression of the components in terms of (as we show, real-valued) functions A–Z (see text). Row 4: Values of functions A–Z at $k = 0$. These values recover hydrodynamic equations up to Burnett approximation and have been obtained analytically from (9) by taking into account lowest order terms in k only. Row 5: Parity with respect to z—symmetric ($\oplus$) or antisymmetric ($\ominus$)—of the part of the corresponding δX entering the averaging in row 2, and whether this part is imaginary or real valued (see Fig. 2). Row 3 is an immediate consequence of row 5.

$\sigma_1^{\|}$	$\sigma_2^{\|}$	$\sigma_3^{\|}$	σ_4
$\langle \lambda^{\|} \delta X_1 \rangle$	$\langle \lambda^{\|} \delta X_2 \rangle$	$\langle \lambda^{\|} \delta X_3 \rangle$	$\langle c_{\|} c_{\perp} \delta Y_4 \rangle$
$-k^2 B$	ikA	$-k^2 C$	ikD
$B_0 = -\frac{1}{3}$	$A_0 = -\frac{2}{3}$	$C_0 = 0$	$D_0 = -\frac{1}{2}$
real, $\oplus$	imag, $\oplus$	real, $\oplus$	imag, $\ominus$
$q_1^{\|}$	$q_2^{\|}$	$q_3^{\|}$	q_4
$\langle \gamma^{\|} \delta X_1 \rangle$	$\langle \gamma^{\|} \delta X_2 \rangle$	$\langle \gamma^{\|} c_{\|} \delta X_3 \rangle$	$\langle (c^2 - \frac{5}{2}) c_{\perp} \delta Y_4 \rangle$
ikX	$-k^2 Z$	ikY	$-k^2 U$
$X_0 = 0$	$Z_0 = -\frac{1}{6}$	$Y_0 = -\frac{5}{4}$	$U_0 = \frac{1}{2}$
imag, $\ominus$	real, $\ominus$	imag, $\ominus$	real, $\oplus$

with plain abbreviations $\tilde{B} \equiv \frac{1}{2} - k^2 B$, $\tilde{C} \equiv \frac{1}{2} - k^2 C$, and $\tilde{Z} \equiv \frac{2}{3}(1 - k^2 Z)$. Note that the checkerboard structure of the matrix $\mathbf{M}$ (8) is particularly useful for studying properties of the hydrodynamic Eqs. (6), such as hyperbolicity and stability (see [7] and below), once the functions A–Z are explicitly evaluated. For that, we still require $\delta \mathbf{X}$. Combining (5) and (6), and requiring that the result holds for any $\mathbf{x}$ (invariance condition), we obtain a closed, singular integral equation (invariance equation) for complex-valued $\delta \mathbf{X}$,

$$\delta \mathbf{X} = \mathbf{X}^0 \cdot (\mathbf{M} + [ikc_{\|} + 1]\mathbf{I})^{-1} - \mathbf{X}^0. \qquad (9)$$

Notice that $\delta \mathbf{X}$ vanishes for $k = 0$. We remind the reader that we use orthogonal basis functions (irreducible moments cf. Table I) to solve (9). The implicit Eq. (9) is identical with the eigenclosure (4), and is our main and practically useful result, with $\mathbf{M}$ from (8), $c_{\|}$, $\mathbf{X}^0$, and A–Z defined in and just before (3) and Table I, respectively.

We iteratively calculate (i) $\delta \mathbf{X}$ directly from (9) for each k in terms of $\mathbf{M}$, (ii) subsequently calculate moments from $\delta \mathbf{X}$ by either symbolical or numerical integration (both approaches produce same results within machine precision, we found simple numerical integration on a regular 500×100 grid in c, z space with grid spacing 0.01 on both axes sufficient to reproduce analytical results). Importantly, the fix point of the iteration (i)-(ii)-(i)-.. is unique for each k, i.e., does not depend on the initial values for moments A–Z, as long as we choose real-valued ones which are consistent with (9), as we prove in the next paragraph. In addition, two other computational strategies
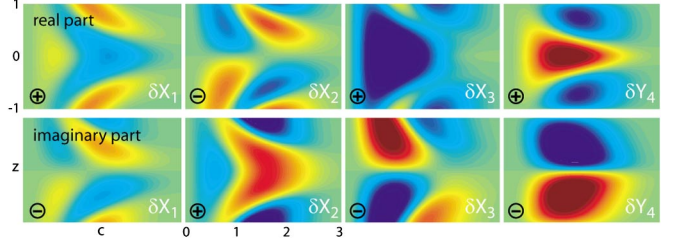

FIG. 2 (color online). Sample distribution function $f(\mathbf{c}, \mathbf{k})$ at $k = 1$, fully characterized by the four quantities $\delta X_{1,2,3}(c, z)$ and $\delta Y_4(c, z)$. Shown here are both their real (top) and imaginary parts (bottom row). In order to improve contrast, we actually plot $\ln|1 + f^{\text{GM}} \delta X_\mu|$ multiplied by the sign of δX_μ. Same color code for all plots, ranging from -0.2 (red) to $+0.2$ (blue).

were implemented: First, we used continuation of functions A–Z from their values at $k = 0$ to solve (9) with an incremental increase of k, where the solution at k was used as the initial guess for $k + dk$. Second, we used also a continuation "backwards" in which the solution at some k (obtained by convergent iterations with a random initial condition) was used as the initial guess for a solution at $k - dk$. Both these strategies returned the same values of functions A–Z as computed by iterations from arbitrary initial condition. The solution $\delta \mathbf{X}$ allows to calculate the whole distribution function f via (4) as illustrated by Fig. 2. For resulting moments for a wide range of k values see Fig. 3.

Finally, we need to clarify the origin of row 5 in Table I (which is directly illustrated by Fig. 2) and its implication on the structure of $\mathbf{M}$ (8) whose entries are—a priori—complex-valued functions to be calculated with complex-valued $\delta \mathbf{X}$. We wish to make use of the fact that all integrals over z vanish for odd integrands. To this end we introduce abbreviations $\oplus$ ($\ominus$) for a real-valued quantity which is even (odd) with respect to the transformation $z \to -z$. One notices $\mathbf{X}^0 = (\oplus, \ominus, \oplus, \oplus)$, and we recall that A–Z are integrals over either even or odd functions in z, times a component of $\delta \mathbf{X}$ (see Table I). Let us prove the consistency of the specified symmetry of $\mathbf{M}$ and the invariance condition: Start by assuming A–Z to be real-valued functions. Then $M_{\mu\nu} = \oplus$ if $\mu + \nu$ is even, and $M_{\mu\nu} = i\oplus$ otherwise. This implies $\delta X_1 = \oplus + i\ominus$, $\delta X_2 = \ominus + i\oplus$, $\delta X_3 = \oplus + i\ominus$, and $\delta X_4 = \oplus + i\oplus$, i.e., different symmetry properties for real and imaginary parts. With these "symmetry" expressions for $\mathbf{X}^0$, $\delta \mathbf{X}$, and $\mathbf{M}$ at hand, we can insert into the right-hand side of the equation, $\delta \mathbf{X} = (\mathbf{X}^0 + \delta \mathbf{X}) \cdot (\mathbf{M} + i\ominus \mathbf{I})$, which is identical with the invariance Eq. (9). There are only two cases to consider, because $\mathbf{M}$ has a checkerboard structure, i.e., only two types of columns: Columns $\mu = 1$ and $\mu = 3$: $\delta X_\mu = \oplus + i\ominus$ because $M_{1-3,4} = 0$; Columns $\mu \in \{2, 4\}$: $\delta X_\mu = \oplus + i\ominus$ if $M_{\mu,1-3} = 0$ (which is the case for column 4) and $\ominus + i\oplus$ if $M_{\mu,4} = 0$ (which is the case for column 2).

We have thus shown that both sides of the invariance Eq. (9) have equal symmetry properties, and that $\delta \mathbf{X}$ with
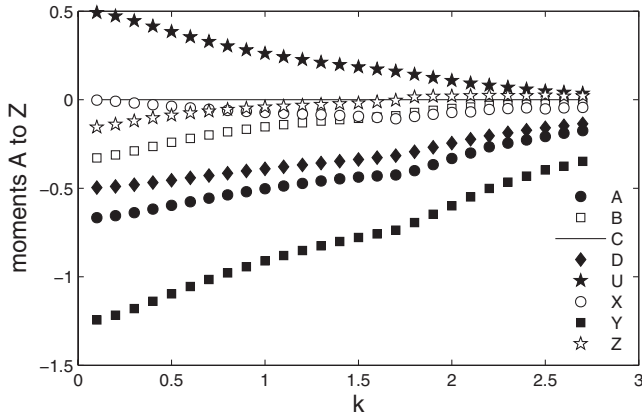
FIG. 3. Moments *A–Z* vs wave number *k* obtained with the solution of (9).

the specified symmetries is consistent with real-valued moments *A–Z*. The proof implies, that any iteratively obtained solution, if it exists, starting with arbitrary real-valued moments *A–Z* in (9) to evaluate $\delta\mathbf{X}$, must converge to real-valued solution *A–Z*. Since the solution is smoothly varying with *k*, and since *A–Z* at *k* = 0 are known and are real-valued, the moments must be real-valued over the whole *k* space.

With the result for the functions *A–Z* at hand, the extended hydrodynamic equations are closed. Let us briefly discuss the pertinent properties of this system. First, the generalized transport coefficients are given by the non-trivial eigenvalues of $-k^{-2}\mathrm{Re}(\mathbf{M})$: $\lambda_2 = -A$ (elongation viscosity), $\lambda_3 = -\frac{2}{3}Y$ (thermal diffusivity), and $\lambda_4 = -D$ (shear viscosity). All these generalized transport coefficients are non-negative (see Fig. 3). Second, computing the eigenvalues of matrix $\mathbf{M}$ we obtain the dispersion relation $\omega(k)$ of the corresponding hydrodynamic modes already presented in Fig. 1. Third, a suitable transform of the hydrodynamic fields, $\mathbf{x}' = \mathbf{T} \cdot \mathbf{x}$, where $\mathbf{T}$ is a real-valued matrix, can be established such that the transformed hydrodynamic equations, $\partial_t \mathbf{x}' = \mathbf{M}' \cdot \mathbf{x}'$, with $\mathbf{M}' = \mathbf{T} \cdot \mathbf{M} \cdot \mathbf{T}^{-1}$ is manifestly hyperbolic and stable; $\mathrm{Im}(\mathbf{M}')$ is symmetric, $\mathrm{Re}(\mathbf{M}')$ is symmetric and nonpositive semi-definite. The corresponding transformation matrix $\mathbf{T}$ can be easily read off the results obtained in [7] for Grad's systems since the structure of the matrix $\mathbf{M}$ (8) is identical to the one studied in [7]. We have explicitly verified that matrix $\mathbf{T}$ [Eqs. (21)–(23) in Ref. [7(a)] and (13) in Ref. [7(b)]] with the functions *A–Z* derived herein is real valued and thus render the transformed hydrodynamic equations manifestly hyperbolic and stable. We note that this result—hyperbolicity of exact hydrodynamic equations—strongly supports a recent suggestion by Bobylev to consider a hyperbolic regularization of the Burnett approximation [11]. Similarly, using the hyperbolicity, an *H* theorem is elementary proven as in [11,7(b)]. Finally, using the accurate data for functions *A–Z*, we can write analytic approximations for the hydrodynamic Eqs. (6) in

such a way that hyperbolicity and stability is not destroyed in such an approximation (see [7(a)]).

In conclusion, we derived exact hydrodynamic equations from the linearized Boltzmann-BGK equation. The main novelty is the numerical nonperturbative procedure to solve the invariance equation. In turn, the highly efficient approach is made possible by choosing a convenient coordinate system and establishing symmetries of the invariance equation. The invariant manifold in the space of distribution functions is thereby completely characterized, that is, not only equations of hydrodynamics are obtained but also the corresponding distribution function is made available. The predicted smoothness and extendibility of the spectrum to all *k* is expected to have some implications for micro-resonators where the quality of the resonator becomes better at very high frequency—that is compatible with our prediction. The damping of all the modes saturates while the imaginary part of the acoustic modes frequency grows. The pertinent data can be used, in particular, as a much needed benchmark for computation-oriented kinetic theories such as lattice Boltzmann models, as well as for constructing novel models [12]. The present approach can be extended to the Boltzmann equation with other collision terms, and supplemented with appropriate boundary conditions cf. [13], to study sound-wave propagation.

---

*il.karlin@lav.mavt.ethz.ch
†URL: www.complexfluids.ethz.ch

[1] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, New York, 1970).
[2] A. Beskok and G. E. Karniadakis, *Microflows: Fundamentals and Simulation* (Springer, Berlin, 2001).
[3] S. Ansumali, I. V. Karlin, S. Arcidiacono, A. Abbas, and N. I. Prasianakis, Phys. Rev. Lett. **98**, 124502 (2007).
[4] D. M. Karabacak, V. Yakhot, and K. L. Ekinci, Phys. Rev. Lett. **98**, 254505 (2007).
[5] A. V. Bobylev, Sov. Phys. Dokl. **27**, 29 (1982).
[6] A. N. Gorban and I. V. Karlin, Phys. Rev. Lett. **77**, 282 (1996).
[7] (a) M. Colangeli, I. V. Karlin, and M. Kröger, Phys. Rev. E **75**, 051204 (2007); (b) **76**, 022201 (2007).
[8] C. Cercignani, *Theory and Application of the Boltzmann Equation* (Scottish Academic Press, Edinburgh, 1975).
[9] A. N. Gorban and I. V. Karlin, Transp. Theory Stat. Phys. **23**, 559 (1994).
[10] M. Kröger, *Models for Polymeric and Anisotropic Liquids* (Springer, Berlin, 2005).
[11] A. V. Bobylev, J. Stat. Phys. **124**, 371 (2006).
[12] S. S. Chikatamarla and I. V. Karlin, Phys. Rev. Lett. **97**, 190601 (2006).
[13] R. D. M. Garcia and C. E. Siewert, Z. Angew. Math. Phys. **57**, 94 (2005).