Wigner Crystallization in Rapidly Rotating 2D Dipolar Fermi Gases

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We study the competition between the Wigner crystal and the Laughlin liquid states in an ultracold quasi-two-dimensional rapidly rotating polarized fermionic dipolar gas, and find that the Wigner crystal has a lower energy below a critical filling factor. We examine the quantum crystal to liquid transition for different confinements in the third direction. Our analysis of the phonon spectra of the Wigner crystal taking into account the phonon-phonon interactions also shows the stability of the Wigner crystal for sufficiently low filling factors ($\nu < 1/7$).

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Recently, remarkable progress has been made in studies of strongly correlated systems of ultra cold gases. A seminal breakthrough was the observation of Mott insulator-superfluid transitions in Bose gases in optical lattices [1], followed by studies of low dimensional gases, fermionic superfluidity or ultracold disordered gases (for recent reviews see [2,3]). A particularly fascinating route toward creation of strongly correlated states is the one that uses rapidly rotating gases. The rotation is formally equivalent to a magnetic field (in the rotational frame), which reorganizes free-particle states into discrete highly degenerate Landau levels. As a result, the properties of the system become very sensitive to interparticle interactions, in an analogy with the fractional quantum Hall effect (FQHE) for electrons [4], that exhibit a variety of strongly correlated states, among which the Laughlin liquid [5], and the Wigner crystal [6] are the most famous. This analogy has been pointed out in the context of ultracold atomic gases with short range interactions in Refs. [7], but an experimental observation of the FQHE in this case is difficult, partially due to the smallness of the energy gap in the spectrum of quasihole excitations. This problem may be overcome in rotating quasi-2D polarized *dipolar* gases [8], in which the long-range tail of the dipole-dipole interaction results in incompressible Laughlin states with gapped excitations. In this sense, rotating dipolar gases can be viewed as the most close neutral analog to electron systems with Coulomb interactions in a strong magnetic field.

The dipolar interactions in a quasi 2D system could also lead to a crystal ground state similar to the electron Wigner crystal. In the case of an electron gas, both in the presence and in the absence of a magnetic field, the Wigner crystal is formed at low densities (filling factors [9]) where the Coulomb interactions ($\sim 1/R$, where R is a typical length scale) dominate over the kinetic energy ($\sim 1/R^2$) cf. [10]. On the contrary, for nonrotating dipolar gases (where interactions scale as $1/R^3$) it was argued [8] that the ground state is a crystal at *high densities*, as shown in Refs. [11] for the case of bosonic dipolar gas. Remarkably, in the case of a rotating dipolar gas, the Wigner crystal phase is expected at *low densities* [8], similar to the electrons in a magnetic field. Dipolar gases, therefore, present a very unique example of the system which, being rotated, behaves similarly to the electrons in the regime of the FQHE in a strong magnetic field, but demonstrates different behavior in a nonrotating case. This opens an unprecedented possibility for the observation and detailed studies of strongly correlated states of the FQHE and quantum phase transitions between them in clean and well-controlled conditions for possible future applications. It is especially appealing in view of the recent experiments with the dipolar Bose condensate of Chromium [12,13], and progress in trapping and cooling of dipolar molecules [14].

In this Letter we demonstrate the existence of a Wigner crystal phase in a rapidly rotating gas of polarized dipolar fermions by comparing it with the competing liquid state. We then examine the stability of the Wigner crystal by incorporating phonon-phonon interactions, and identify the border of stability (spinodal). It appears that the Wigner crystal is the stable ground state of the systems for low filling factors, typically for $\nu < 1/7$.

We consider a gas of fermionic polarized dipolar particles with a dipole moment d in a rotating cylindrical trap, tightly confined and polarized along the axis of rotation z. For a sufficiently strong confinement in the axial direction, all particles are in the ground state of the axial motion, and the many-body wave function has the form

$$\psi_{3D}(\{\mathbf{r}_i, \zeta_i\}) = \psi_{2D}(\{\mathbf{r}_i\})(l\sqrt{\pi})^{-N/2} \exp\left(-\sum_{i=1}^{N} \zeta_i^2/2l^2\right),$$

where $\mathbf{r}_i = (x, y)$ and ζ_i are radial and axial coordinates of the particles, respectively, l is the extension in the axial direction, N is the total number of particles, and ψ_{2D} is the wave function of the system in the xy plane. The confinement in the axial direction results in an effective interpar-

ticle interaction $v_{2D}(r)$ in the 2D plane:

$$v_{2D}(r) = \frac{d^2}{\pi^2 l^4}\int d\zeta_1 d\zeta_2 \frac{R_{12}^2 - 3\zeta_{12}^2}{R_{12}^5}\exp\left(-\frac{\zeta_1^2+\zeta_2^2}{l^2}\right) = \frac{d^2}{l^3\sqrt{2\pi}}\int_0^\infty d\xi \sqrt{\frac{\xi}{(\xi+1)^3}}\exp\left(-\xi\frac{r^2}{2l^2}\right),$$

where r is the distance between two dipoles in the $2D$ plane, $\zeta_{12} = \zeta_1 - \zeta_2$, and $R_{12} = \sqrt{r^2+\zeta_{12}^2}$. For $r \gg l$ the potential is $v_{2D} \approx d^2/r^3$, while for $r < l$ the effective potential increases logarithmically, $v_{2D} \approx (d^2/l^3)\times\sqrt{2/\pi}\ln(l/r)$. The wave function ψ_{2D} obeys the Schrödinger equation with the effective 2D Hamiltonian in the rotational frame:

$$H = \sum_i\left(-\frac{\hbar^2}{2m}\Delta_i + \frac{m}{2}\omega_\perp^2\mathbf{r}_i^2\right) - \Omega L_z + V_D, \quad (1)$$

where Ω is the rotational frequency, $\omega_\perp$ is the radial trap frequency, $V_D = \sum_{i<j} v_{2D}(|\mathbf{r}_i - \mathbf{r}_j|)$ describes the dipole-dipole interparticle interaction, and L_z is the z component of the total angular momentum. The Hamiltonian (1) can be rewritten in the form

$$H = \sum_i \frac{1}{2m}(-i\hbar\nabla_i - \mathbf{A}_i)^2 - (\Omega - \omega_\perp)L_z + V_D \quad (2)$$

with $\mathbf{A}_i = m\omega_\perp \mathbf{e}_z \times \mathbf{r}_i$ and, therefore, it formally describes a system of charged particles in a constant magnetic field with the cyclotron frequency $\omega_c = 2\omega_\perp$. For noninteracting particles ($V_D = 0$) in the regime of the critical rotation, $\omega_\perp = \Omega$, the properties of the Hamiltonian (2) are well known: the spectrum consists of highly degenerate levels $E_n = \hbar\omega_c(n + 1/2)$ called Landau levels. In the following, we restrict ourselves to a system of particles occupying only the lowest Landau level.

The wave function of a noncorrelated Wigner crystal on the lowest Landau level is [15]:

$$\Psi_C(\{z_i\}) \sim \mathcal{A}\prod_i \exp\left[\frac{-1}{4l_0^2}(|z_i - R_i|^2 + z_i R_i^* - z_i^* R_i)\right], \quad (3)$$

where $z_i = x_i + iy_i$ is the complex representation of the 2D vector $\mathbf{r}_i$, R_i the (complex) lattice site, $l_0 = \sqrt{\hbar/m\omega_c}$ the magnetic length, and $\mathcal{A}$ denotes the antisymmetrization over z_i, that can be omitted for sufficiently low filling factors $\nu = 2\pi l_0^2 n$. One can check that similar to the case of classical dipoles, the energy is minimal for a triangular lattice with particles centered at positions $\mathbf{R}_i = l_1\mathbf{b}_1 + l_2\mathbf{b}_2$, where $l_{1,2}$ are integers, $\mathbf{b}_1 = a(0,1)$, $\mathbf{b}_2 = a(\sqrt{3},1)/2$, and a is the lattice constant determined by the density n of the gas $a^2 = 2/\sqrt{3}n = 4\pi l_0^2/\sqrt{3}\nu$. The energy of the crystal state equals

$$U_C = \langle\Psi_C|V_D|\Psi_C\rangle \approx d^2 n^{3/2}(0.2823 + 0.2146\beta + 0.3388\beta^2 + 0.7456\beta^3 + 2.0676\beta^4 + \ldots),$$

where $\beta = \pi n(2l_0^2 - l^2)$ and the first term corresponds to the energy of the crystal of classical pointlike dipoles, $E_{cl} = \sum_{i<j} d^2/|l_1\mathbf{b}_1 + l_2\mathbf{b}_2|^3 = 5.513 d^2/a^3$.

Competing liquid states can be represented by fermionic Laughlin states

$$\Psi_L(\{z_i\}) \sim \prod_{i<j}(z_1 - z_j)^M \exp\left(-\sum_i |z_i|^2/4l_0^2\right), \quad (4)$$

characterized by an odd integer $M = 1/\nu$ (recent computations for a system with small numbers of dipolar particles [16] show a remarkable overlap of the exact ground state with the Laughlin state). The energy is $U_L = (\nu/2)\times\int_0^\infty r g(r) v_{2D}(r) dr$, where $g(r)$ is the pair correlation function that can be calculated using standard 2D plasma analogy [5]. We calculated $g(r)$ and U_L for all odd M from 1 to 19 for a gas of 512 particles to guarantee the required accuracy.

The comparison of the energies of the Wigner crystal and of the Laughlin liquid for different filling factors ν and extensions l in the axial direction are shown in Fig. 1. We see that below some critical value ν_c, which depends on l, the Wigner crystal has a lower energy and, therefore, for $\nu < \nu_c$ the ground state is expected to be a crystal. Note that in calculating the energies we use the simplest trial wave functions for liquid and crystal states. A better estimate could follow from considering the wave functions of quantum Hall liquids of composite fermions [17] with the filling factors, which are closer to the critical one than $1/M$, and of the correlated Wigner crystals [18]. We, however, do not expect a significant change of our result because at low densities the used wave functions already

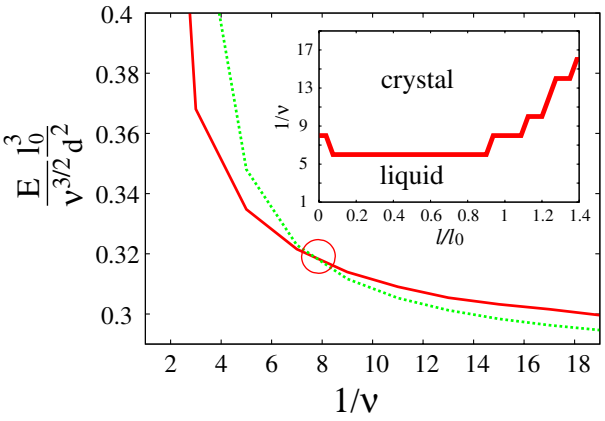

FIG. 1 (color online). Energy per particle for the Wigner crystal (dotted line) and for the Laughlin liquid (solid line) as a function $1/\nu$ for $l = 0$. The inset shows the critical filling factor as a function of the extension in the z direction.

take into account the most important effects of the interparticle correlations. For the same reason we could expect similar behavior from a bosonic dipolar gas. To support the above picture, we demonstrate the stability of the crystal at small ν and approach the liquid-crystal transition from the crystal phase.

We analyze the stability of the crystal phase by considering its phonon spectrum taking into account the phonon-phonon interactions (anharmonicity effects). The appearance of purely imaginary phonon frequencies will indicate the instability of the crystal and, therefore, the transition to a liquid state. Note that this does not actually determine the transition point, rather establishes the upper bound (spinodal) for the crystal phase.

In the harmonic approximation the phonon eigenfrequencies can be obtained from the dynamic equations for displacements $u_{\alpha l}$ of particles along the α-axis from their equilibrium positions $\mathbf{R}_l$ in the lattice [19]

$$m\ddot{u}_{\alpha l} = \sum_{\beta l'} \Phi^{(2)}_{\alpha l,\beta l'} u_{\beta l'} + \omega_c \varepsilon_{\alpha\beta} \dot{u}_{\beta l}, \tag{5}$$

where $\Phi^{(2)}_{\alpha l,\beta l'} = \partial^2 U_C / \partial R_{\alpha l} \partial R_{\beta l'}$ is the dynamical matrix, α, $\beta = x$, y, and $\varepsilon_{\alpha\beta}$ is the antisymmetric tensor, $\varepsilon_{xy} = 1$. The last term in Eq. (5) corresponds to the Coriolis (Lorentz) force due to the rotation (magnetic field).

Without rotation, $\omega_c = 0$, Eq. (5) in the quasimomentum representation reads: $\omega^2 \tilde{u}_\alpha(\mathbf{k}) = \sum_\beta F_{\alpha\beta}(\mathbf{k}) \tilde{u}_\beta(\mathbf{k})$, where $mF_{\alpha\alpha'}(\mathbf{k}) = \sum_l \exp[-i\mathbf{k}(\mathbf{R}_\mathbf{l} - \mathbf{R}_{l'})] \Phi^{(2)}_{\alpha l,\beta l'}$ is the Fourier transform of $\Phi^{(2)}_{\alpha l,\beta l'}$, and the eigenvalues $\omega_s^2(\mathbf{k})$ of $F_{\alpha\alpha'}(\mathbf{k})$ determine the frequencies of the transversal ($s = T$) and longitudinal ($s = L$) phonons. The phonon frequencies for $l = 0$ are shown in Fig. 2. They are linear in k for $k \ll a^{-1}$: $\omega_T \approx (3/\sqrt{8})\omega_d a k$ and $\omega_L \approx \sqrt{11}\omega_T$, where $\omega_d = \sqrt{d^2/ma^5}$ sets the typical value for the frequency of phonons. For a rotating crystal, $\omega_c \neq 0$, the phonon spectrum becomes [19,20]

$$\omega_\pm^2 = \frac{\omega_L^2 + \omega_T^2 + \omega_c^2}{2} \pm \frac{1}{2}\sqrt{(\omega_L^2 + \omega_T^2 + \omega_c^2)^2 - 4\omega_L^2\omega_T^2}.$$

In the case $\omega_c \gg \omega_s$, one has $\omega_+ \approx \omega_c$ and $\omega_- \approx \omega_L \omega_T / \omega_c$.

Higher order (anharmonic) terms in the expansion of the energy of the crystal with respect to the displacements of particles from their equilibrium positions result in phonon-phonon interactions and, therefore, in the renormalization of the phonon frequencies. At a given quasimomentum $\mathbf{k}$, the renormalized frequencies correspond to the poles of the Fourier transform $G_{\alpha\beta}(\omega, \mathbf{k})$ of the phonon Green function $G_{\alpha\beta}(t, \boldsymbol{l} - \boldsymbol{l}) = i[\theta(t)\langle u_{\alpha l}(t) u_{\beta l'}(0)\rangle + \theta(-t)\langle u_{\beta l'}(0) u_{\alpha l}(t)\rangle]_{l'}$ with $\theta(t)$ being the step function (for more details on Green functions for phonons in crystals see, e.g., [21]). The Green function in the harmonic approximation is $G^{(0)-1}_{\alpha\beta}(\omega, \mathbf{k}) = \{\sum_{s=T,L} \mathcal{M}^{(s)}_{\alpha\beta}[\omega_s^2(\mathbf{k}) - \omega^2] - i\varepsilon_{\alpha\beta}\omega_c\omega\} m/\hbar$, where $\mathcal{M}^{(s)}_{\alpha\beta} = e^{(s)}_\alpha e^{(s)}_\beta$ is the projector to the eigenmode s with the polarization $\mathbf{e}^{(s)}$, and the poles of $G^{(0)}_{\alpha\beta}$ are at $\omega = \omega_\pm(\mathbf{k})$. In the presence of phonon-phonon interactions, the Green function obeys the Dyson equation $G^{-1}_{\alpha\beta}(\omega, \mathbf{k}) = G^{(0)-1}_{\alpha\beta}(\omega, \mathbf{k}) - \Sigma_{a\beta}(\omega, \mathbf{k})$, where the phonon self-energy function Σ incorporates all the effects of phonon-phonon interactions.

We calculate the phonon self-energy in the one-loop approximation (the validity of this approximation was discussed in Refs. [22–24]). This corresponds to taking into account only the fourth $\Phi^{(4)} \sim \partial^4 U_C / \partial R^4$ and the square of the third $\Phi^{(3)} \sim \partial^3 U_C / \partial R^3$ order anharmonic terms; see Fig. 3. We then solve the Dyson equation numerically by successive iterations until we either obtain a self-consistent solution or the iteration breaks down due to the appearance of purely imaginary phonon frequencies. At very low filling factors the phonon-phonon interactions do not play any significant role and the system behaves harmonically. However, the effects of anharmonicity become progressively important with increasing the filling factor, and, finally, at some critical filling factor ν_c there appear purely imaginary frequencies signaling the instability of the crystal. We always observe this phonon instability for $k \to 0$. This indicates the breakdown of the crystalline order and

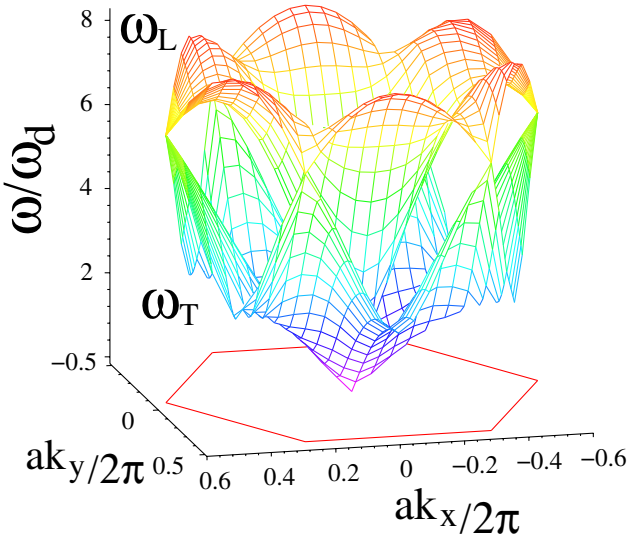

FIG. 2 (color online). Energy of the transverse ω_T (lower surface) and longitudinal ω_L phonons.

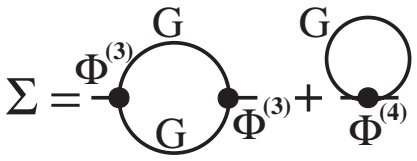

FIG. 3. One-loop diagrams for the phonon self-energy.

rules out any possible structural phase transition. Therefore, ν_c sets the upper bound for the stability of the crystal. The critical filling factor depends on the confinement l in axial direction as well as on the ratio of ω_d/ω_c that measures the strength of the dipole-dipole interaction relative to the Landau level spacing. For $l = 0$ we find $\nu_c^{-1} = 4.33\exp(-0.0021\omega_c/\omega_d) + 5.77$. Note that for $\omega_c \gg \omega_d$ the critical filling factor ν_c becomes insensitive to the strength of the interparticle interaction. In this limit the Wigner crystal is stable for $\nu < \nu_c = 0.174$, which is in a good agreement with our previous energetic consideration for the tight confinement along the z axis, $l < l_0$. We also calculate the Lindemann parameter $\gamma = \sqrt{\langle \mathbf{u}^2 \rangle}/a$—the ratio of the average displacement of a particle in the lattice from its equilibrium position to the lattice spacing a. For $\omega_c \gg \omega_d$ we find $\gamma = 0.28$ that is within the range of values of the Lindemann parameter for various 2D crystals.

We see that the filling factor is the critical parameter that controls the ground state of the system. The way of manipulating the filling factor in experiments with rotating gases depends on an experimental setup. In the case of a critical rotation with an extra (quartic) confinement [25] this can be achieved by changing the number of particles (the size of the system is fixed by an extra confinement). For a purely harmonic confinement and "under critical" rotations [26,27], the filling factor can be changed by varying the difference $\Omega - \omega_\perp$. In this setup, the size of the system and, therefore, the filling factor is determined by a competition between the interparticle interaction V_{D} and the "tilting" term $(\Omega - \omega_\perp)L_z$ [see Eq. (2)]. In both cases, the appearance of a crystal order could be detected by studying the shot noise correlations using the Hanbury Brown–Twiss effect [28] in a similar way as it was used to observe the Mott insulator-superfluid transition in a lattice Bose gas [1].

In conclusion, we show that a rapidly rotating polarized 2D dipolar gas undergoes a transition from a liquid into a crystal when the filling factor becomes less that some critical value (typically for $\nu < 1/7$). This shows that rapidly rotating polarized dipoles behave in a way similar to electrons in a strong magnetic field (regime of the quantum Hall effect), and thus provide a new experimental possibility for studying the fractional Hall effect in a completely different system, as well as creating strongly correlated states useful for various applications.

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