Macroscopic Resonant Tunneling in the Presence of Low Frequency Noise

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We develop a theory of macroscopic resonant tunneling of flux in a double-well potential in the presence of realistic flux noise with a significant low-frequency component. The rate of incoherent flux tunneling between the wells exhibits resonant peaks, the shape and position of which reflect qualitative features of the noise, and can thus serve as a diagnostic tool for studying the low-frequency flux noise in SQUID qubits. We show, in particular, that the noise-induced renormalization of the first resonant peak provides direct information on the temperature of the noise source and the strength of its quantum component.

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Superconducting qubits based on different forms of quantum dynamics of magnetic flux in SQUIDs continue to demonstrate steady progress—see, e.g., [1–3]. Nevertheless, the low-frequency flux noise in SQUID structures still provides the major obstacle to their further development to the level necessary for quantum computing applications. Because of this noise, quantum coherence is essentially reduced to the states with the same average flux (e.g., to qubit operation at the "optimal point") [1,3,4]. Only very limited coherence exists between the states which differ by their flux values [2]. In superconducting charge qubits, the low-frequency charge noise is most probably produced by elementary charges tunneling between impurity states localized in the insulator parts of the structure [5]. In contrast to this, it is more difficult to construct a possible model of the low-frequency noise of magnetic flux. Elementary magnetic moments of nuclei or electrons [6] are small and require very large concentration of impurity states to explain the typical value $\delta\Phi \sim 0.1\ \mathrm{m}\Phi_0$ of the observed noise [7,8]. So the origin of the low-frequency flux noise in SQUID structures remains not fully understood, making it interesting to probe its properties in different experiments. In this work, we show that one of the interesting dynamic processes in SQUIDs, macroscopic resonant tunneling of flux between the wells of the double-well potential [9,10] can serve as noise diagnostic tool. In particular, the shape and position of the resonant flux tunneling peaks reflects the temperature and the strength of quantum component of the low-frequency noise.

Specifically, we consider a SQUID in the regime when its flux potential has the double-well form (Fig. 1). The extraction of precise parameters of the system, e.g., the spectrum ε_j of the energy levels in the two wells, requires its direct numerical simulation. Considerable insight into the flux dynamics in this system can still be gained from analytical treatment based on the general features of the spectrum. The states localized in each well are separated by large energy intervals on the order of the plasma frequency ω_p, while the tunnel coupling Δ between the states in the opposite wells is small ($\hbar = k_B = 1$), $\Delta \ll \omega_p$. This means that the rate of flux transfer between the two wells exhibits resonant peaks as a function of bias between the wells [9] whenever the energy distance ϵ between a pair of levels in the opposite wells becomes small, $\epsilon \ll \omega_p$. In the vicinity of such a resonance, the influence of the environment on the flux dynamics in this system of levels separates into two different effects [10] even if the dissipation is produced by physically one and the same environment. The first is "intrawell" relaxation that causes the dissipative transitions within each well, and is sensitive to the properties of environment at large energies $\sim\omega_p$, and the second is "interwell" relaxation associated with relaxation dynamics within the two tunnel-coupled states, and sensitive to the environment properties at low energies $\ll \omega_p$. In terms of the coupling of flux to the environment, interwell relaxation is dominantly affected by fluctuations of the bias ϵ that shift one well of the double-well system as a whole relative to the other. The purpose of this work is to extend the previous theories [10,11] of the resonant flux tunneling to the regime of strong interwell relaxation. This regime is made experimentally relevant by apparently unavoidable low-frequency flux noise in the SQUID structures. The developed approach should also be valid for the description of tunneling in other double-well systems besides flux qubits.

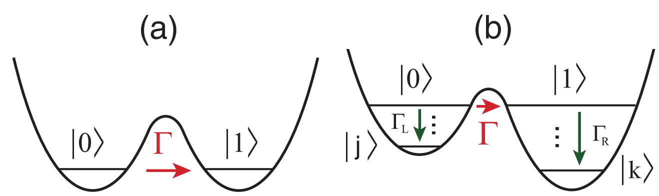

FIG. 1 (color online). Schematic energy diagram of the macroscopic resonant tunneling in the double-well system between (a) two lowest energy states and (b) two excited states in each well.

We start by calculating the transition rate for tunneling between the lowest energy levels $|0\rangle$ and $|1\rangle$ in the "left" and "right" wells, respectively, [Fig. 1(a)]. Such a process can be described by a two-state model: $H = H_S + H_B + H_{\rm int}$, where H_B is the environment's Hamiltonian and

$$H_S = -\tfrac{1}{2}(\Delta\sigma_x + \epsilon\sigma_z), \qquad H_{\rm int} = -\tfrac{1}{2}\sigma_z Q \tag{1}$$

are the system and interaction Hamiltonians, respectively. Here, $\sigma_{x,z}$ are the Pauli matrices and Q is an operator acting on the environment. We adopt the Gaussian approximation for which the dissipative dynamics is characterized completely by the spectral properties of the reservoir force Q; hence, the reservoir Hamiltonian H_B does not need to be made explicit. If the environment is in equilibrium at temperature T, the (unsymmetrized) spectral density of the noise Q, defined as

$$S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt e^{i\omega t}\langle Q(t)Q(0)\rangle,$$

where $\langle\ldots\rangle \equiv {\rm Tr}_B\{\rho\ldots\}$, can be expressed in the usual way in the basis of energy eigenstates $|n\rangle$ of H_B with eigenvalues $\mathcal{E}_n$ and equilibrium probability density ρ_n

$$S(\omega) = \sum_{nm}\rho_n|\langle n|Q|m\rangle|^2\delta(\mathcal{E}_n - \mathcal{E}_m + \omega). \tag{2}$$

We find the rate Γ of flux tunneling between the resonant states $|0\rangle$ and $|1\rangle$ in the incoherent regime $\Delta \ll W$, where W is the noise-induced resonance width. In this case, Γ can be calculated in the lowest-order perturbation theory in the tunneling term $V = \Delta\sigma_x/2$ in the Hamiltonian (1). We assume that the width W is still small on the scale of the plasma energy ω_p, so that the system dynamics can be reduced to the two resonant levels even in the presence of noise. This condition is satisfied in typical experiments, where W is on the order of a few GHz, while $\omega_p \simeq 30$ GHz [9,12]. The lowest-order transition rate between the states $|i\rangle$ and $|f\rangle$ can be obtained using Fermi golden rule:

$$\begin{aligned}\Gamma_{i\to f} &= 2\pi|\langle i|V|f\rangle|^2\delta(E_i - E_f)\\ &= \int_{-\infty}^{\infty} dt e^{i(E_i-E_f)t}|\langle i|V|f\rangle|^2\\ &= 2\,{\rm Re}\int_0^{\infty} dt\langle i,t|V|f,t\rangle\langle f,0|V|i,0\rangle.\end{aligned} \tag{3}$$

where $E_{i,f}$ are the eigenvalues of the *unperturbed* total (system + environment) Hamiltonian for states $|i\rangle$, $|f\rangle$. In our case, these states are the systems resonant flux states $|0\rangle$ and $|1\rangle$ plus the states of the environment at the beginning and the end of the transition. The time evolution of these states can be written as ($\alpha = i, f$):

$$|\alpha, t\rangle = U(t)e^{i\epsilon\sigma_z t/2}|\alpha, 0\rangle. \tag{4}$$

Here, $U(t) = \mathcal{T}\exp\{(i/2)\sigma_z\int_{-\infty}^{t} Q(\tau)d\tau\}$ is the evolution operator in the interaction representation, with $\mathcal{T}$ denoting the time ordering and $Q(t) = e^{iH_Bt}Qe^{-iH_Bt}$. To avoid description of the process of "switching on" of in general strong interaction with the low-frequency environment, we extended the starting time of the evolution $U(t)$ to $-\infty$. Substituting (4) into (3), we find $\Gamma_{0\to1}(\epsilon) = \Gamma_{1\to0}(-\epsilon) \equiv \Gamma(\epsilon)$, with

$$\Gamma(\epsilon) = \frac{\Delta^2}{2}\,{\rm Re}\int_0^{\infty} dt e^{i\epsilon t}\langle U_-^\dagger(t)U_+(t)U_+^\dagger(0)U_-(0)\rangle, \tag{5}$$

where $U_\pm = \mathcal{T}\exp\{\pm(i/2)\int_{-\infty}^{t} Q(\tau)d\tau\}$. Here, we have summed the rates $\Gamma_{i\to f}$ over the initial (with equilibrium density matrix ρ) and final states of the environment. The correlator in Eq. (5) can be calculated in the Gaussian approximation by expanding each of the operators $U_\pm$ up to the second order in Q, averaging the result and exponentiating it back. This gives

$$\begin{aligned}&\langle U_-^\dagger(t)U_+(t)U_+^\dagger(0)U_-(0)\rangle\\ &\quad = \exp\left\{\int_0^t d\tau\int_{-\infty}^{0} d\tau'\langle Q(\tau)Q(\tau')\rangle\right\}.\end{aligned}$$

Expressing the noise correlator in this equation through the spectral density (2), we reduce Eq. (5) to

$$\Gamma(\epsilon) = \frac{\Delta^2}{2}\,{\rm Re}\int_0^{\infty} dt e^{i\epsilon t}\exp\left\{\int d\omega S(\omega)\frac{e^{-i\omega t}-1}{\omega^2}\right\}. \tag{6}$$

The integral over time in Eq. (6) converges on the scale W^{-1} which defines the width W of the resonance. The line shape of the resonance is determined therefore by the noise frequencies $\omega < W$. If $S(\omega)$ is essentially constant in this frequency range, $S(\omega) = S(0)$, i.e., if the noise is sufficiently broadband and/or weak, the tunneling rate (6) has Lorentzian line shape

$$\Gamma = \frac{1}{2}\frac{\Delta^2 W}{\epsilon^2 + W^2}, \qquad W = \pi S(0). \tag{7}$$

In general, Eq. (6) can lead to line shapes that are not Lorentzian. If the cutoff frequency ω_c of the low-frequency part of the noise satisfies the condition $\omega_c \ll W$ (i.e., the noise is strong and has low-frequency), then one can expand the exponent $e^{-i\omega t}$ up to the second order in ωt to obtain

$$\Gamma(\epsilon) = \sqrt{\frac{\pi}{8}}\frac{\Delta^2}{W}\exp\left\{-\frac{(\epsilon-\epsilon_p)^2}{2W^2}\right\}, \tag{8}$$

$$W^2 = \int d\omega S(\omega), \qquad \epsilon_p = \mathcal{P}\int d\omega\frac{S(\omega)}{\omega}. \tag{9}$$

We assume that both integrals in (9) are finite, with the second integral being understood as a principle value. We see that, as usual, the noise with a significant low-frequency part yields a Gaussian line shape, in contrast to the Lorentzian line shape (7) in the case of the broadband noise. Equation (8) shows also that, in general, the low-frequency noise not only broadens the resonance but also shifts it to the nonvanishing bias $\epsilon \simeq \epsilon_p$. This means that effectively the first resonant peak splits in two, described by $\Gamma(\epsilon)$ and $\Gamma(-\epsilon)$, for the two different directions of tunneling.

If the environmental source of the low-frequency noise is in equilibrium at temperature T, then symmetric and antisymmetric (in frequency) parts of the noise intensity, $S_{\pm}(\omega) = [S(\omega) \pm S(-\omega)]/2$, are related by the fluctuation-dissipation theorem: $S_{-}(\omega) = S_{+}(\omega) \times \tanh(\omega/2T)$. The antisymmetric noise gives also the dissipative part $\sigma(\omega, T)$ of the linear response (e.g., conductance) of the environment generating the noise: $S_{-}(\omega) = \omega\sigma(\omega, T)$. Since in (9) the width W and shift ϵ_p of the Gaussian tunneling peak are determined, respectively, by symmetric and antisymmetric parts of the noise: $W^2 = \int d\omega S_{+}(\omega)$ and $\epsilon_p = \int d\omega S_{-}(\omega)/\omega$, the fluctuation-dissipation theorem relates W and ϵ_p. This relation is particularly simple in the case of low-frequency noise, when all the relevant frequencies, $\omega \leq \omega_c$, are small on the scale of temperature T: $\omega \ll T$. For typical cutoff frequencies on the order GHz, this condition holds even at sub-Kelvin temperatures characteristic for experiments with solid-state qubits (1 GHz $\simeq$ 50 mK). In this case, $\tanh(\omega/2T) \simeq \omega/2T$, yielding

$$W^2 = 2T\epsilon_p. \tag{10}$$

Equation (10) is one of the main conclusions of the theory developed in this work. It shows that, qualitatively, nonvanishing splitting $2\epsilon_p$ of the two parts of the first resonant peak is a manifestation of the finite quantum component of the low-frequency flux noise. In the regime of classical noise (which is valid, due to the fluctuation-dissipation theorem, only as a high-temperature approximation), the spectrum $S(\omega)$ is symmetric in ω and $\epsilon_p = 0$. Quantitatively, this classical limit is reached at $T \gg W$, when $\epsilon_p \ll W$ and the two peaks merge.

The above method of the calculation of the tunneling rates can be generalized to the case of resonant tunneling between two higher energy levels in the wells. In that case, in addition to the interwell relaxation process, we will also have intrawell relaxation. We still use $|0\rangle$ and $|1\rangle$ to denote the two resonant flux states, which can now be excited states in their corresponding wells [see Fig. 1(b)]. In addition to Hamiltonians (1), we need to add to the total Hamiltonian two other terms: $H_0 = \sum_{k\neq 0,1} \varepsilon_k |k\rangle\langle k|$ that describes the energy of the flux energy eigenstates $|k\rangle$ with $k \neq 0, 1$, and

$$H_r = \sum_{k\neq k'} \phi_{kk'}(|k\rangle\langle k'| + |k'\rangle\langle k|)Q. \tag{11}$$

that produces the intrawell relaxation. The sum in H_r goes over the pairs of states $|k\rangle$ and $|k'\rangle$ that belong to the *same* well and $\phi_{kk'}$ are the coupling matrix elements between these states that drive the relaxation transitions.

We make two main assumptions about the properties of decoherence. First is that the high-frequency component of the noise is sufficiently weak, so that the intrawell relaxation rates in the two wells are small on the scale of ω_p, $\Gamma_{L,R} \ll \omega_p$, so that it makes sense to discuss resonances in the tunneling rate. The relaxation rates can still be large on the scale of tunneling strength Δ. Under a natural assumption $T \ll \omega_p$, effects of the weak (on the scale of ω_p) intrawell relaxation depend only on the spectral components of the $\omega \simeq \omega_p$ that enter into the relaxation rates $\Gamma_{L,R}$, and we can characterize them directly by these rates.

We find the rate Γ of flux tunneling between the resonant states $|0\rangle$ and $|1\rangle$ the same way as before. Since the environmental modes that contribute to the interwell and intrawell relaxations are from different parts of the environment spectrum and the coupling Hamiltonians $H_{\rm int}$ and H_r commute (the matrix elements $\phi_{kk'}$ are nonvanishing only between the states within the same well), Eq. (4) can be written as direct combination of the evolution due to the high-frequency noise $\tilde{U}(t) = \mathcal{T}\exp\{-i\int_0^t H_r(\tau)d\tau\}$ and the low-frequency noise $U(t)$, as defined below (4):

$$|\alpha, t\rangle = \tilde{U}(t)U(t)e^{i\epsilon\sigma_z t}|\alpha, 0\rangle,$$

where $H_r(t) = e^{i(H_B+H_0)t}H_r e^{-i(H_B+H_0)t}$. Since the relaxation is assumed weak on the scale of the plasma frequency, such a trace over the high-frequency part of the environment can be done in the relaxation approximation, when the relaxation dynamics is determined by the transition of the lowest (second) order in H_r that are characterized by the relaxation rates $\Gamma_{L,R}$. Then,

$$\mathrm{Tr}_B\{\rho\langle j|\tilde{U}^\dagger(t)|j\rangle\langle j'|\tilde{U}(t)|j'\rangle\} = e^{-\gamma t}, \tag{12}$$

where j, j' denote the two opposite resonant flux states coupled by tunneling: $j, j' \in \{0, 1\}$, $j \neq j'$, and $\gamma \equiv (\Gamma_L + \Gamma_R)/2$. Equation (12) is written in the form convenient for the calculation of the tunneling rate in Eq. (3). It can be understood more directly if one notices that it gives the time evolution of the off-diagonal element $|j\rangle\langle j'|$ of the flux density matrix averaged over the environment. According to the standard theory of weak relaxation (see, e.g., [13]), such a matrix element decays with the rate equal to the half-sum of all the relaxation rates out of the states $|j\rangle$ and $|j'\rangle$—cf. Eq. (12). After the high-frequency part of the environment is traced out, expression for the flux tunneling rate takes the form

$$\Gamma(\epsilon) = \frac{\Delta^2}{2}\,\mathrm{Re}\int_0^\infty dt e^{(i\epsilon-\gamma)t}\exp\left\{\int d\omega S(\omega)\frac{e^{-i\omega t}-1}{\omega^2}\right\}. \tag{13}$$

This equation describes higher resonant peaks, when the flux tunnels from the ground state in the initial (e.g., left) well into an excited state of the target (right) well. In this case, $\gamma = \Gamma_R/2$, and the line shape of the resonance (13) is determined by the combined action of the relaxation broadening and the low-frequency noise. If the noise is strong, using the same approximation as for the first resonance, we get

$$\Gamma(\epsilon) = \sqrt{\frac{\pi}{8}}\frac{\Delta^2}{W}\,\mathrm{Re}\left[w\left(\frac{\epsilon - \epsilon_p + i\gamma}{\sqrt{2}W}\right)\right], \qquad (14)$$

where the parameters are given as before by Eq. (9), and $w(x)$ is the complex error function defined as

$$w(x) = e^{-x^2}[1 - \mathrm{erf}(-ix)] = e^{-x^2}\frac{2}{\sqrt{\pi}}\int_{-ix}^{\infty} e^{-t^2}dt.$$

Equation (14) describes the transition between the Gaussian and Lorentzian line shapes in the noise- ($W \gg \gamma$) and the relaxation-dominated ($W \ll \gamma$) regimes, respectively. Independently of this, if the noise is quantum and $\epsilon_p \neq 0$, the higher resonant peaks are also shifted by ϵ_p away from their positions determined by the energy levels in the well that follow from the Schrödinger equation for the flux dynamics in the absence of environment.

Finally, we consider the microwave-activated tunneling between the two wells. For the simplest case where the tunneling occurs between the two lowest energy levels of each well, the process can be described by the Hamiltonian $H_{rf} = -\epsilon\sigma_z + \hat{V}\cos\Omega t$, where $\hat{V}$ is an operator that couples microwaves to the flux dynamics. Unitary transformation $U = e^{-i(\Omega t/2)\sigma_z}$ changes this Hamiltonian into $\tilde{H}_{rf} = UH_S U^\dagger + i\dot{U}U^\dagger$, and gives in the rotating wave approximation: $\tilde{H}_{rf} = -\frac{1}{2}(\tilde{\epsilon}\sigma_z + \tilde{V}\sigma_x)$, where $\tilde{\epsilon} = \epsilon - \Omega \ll \Omega$ is the detuning of microwave radiation and $\tilde{V} = \langle 0|\hat{V}|1\rangle$. In the absence of the environment, the system would undergo Rabi oscillations with the Rabi frequency $\Omega_R = \sqrt{\tilde{\epsilon}^2 + |\tilde{V}|^2}$. If coupling to the environment is strong enough so that $\tilde{V} \ll W$, then the tunneling becomes incoherent with the rate given by (8), but with ϵ and Δ replaced by $\tilde{\epsilon}$ and $|\tilde{V}|$, respectively.

In another interesting regime, the microwave excites the system from the ground state $|k\rangle$ to an excited state $|0\rangle$ in the first well, and it tunnels then to state $|1\rangle$ in the target well (Fig. 1). We make use of the same assumption of small tunnel coupling Δ that can be treated as perturbation. In addition to previously used condition $\Delta \ll W$, this assumption implies also that the tunneling rate $\Gamma(\epsilon)$ is small compared to the rates of excitation or relaxation processes within each well. The balance between excitation and relaxation (Γ_L) in the well establishes the stationary occupation p_0 of the state $|0\rangle$. The standard solution for the relaxation dynamics of the density matrix of the system (see, e.g., [10]) gives $p_0 = |\tilde{V}|^2/(\tilde{\epsilon}^2 + 2|\tilde{V}|^2 + \Gamma_L^2/4)$. Here, again, $\tilde{\epsilon} = \epsilon_0 - \epsilon_k - \Omega$ is the detuning energy, and $\tilde{V} = \langle 0|\hat{V}|k\rangle$. At this stage, one can repeat the same steps in the calculation of the incoherent tunneling rate between the states $|0\rangle$ and $|1\rangle$ as above, and obtain the rate $\tilde{\Gamma} = p_0\Gamma$, where Γ is given by (14), but now with the relaxation broadening determined by the relaxation in both wells, $\gamma = (\Gamma_L + \Gamma_R)/2$. Qualitatively, as before, the shape of the resonant peak of $\tilde{\Gamma}$ is determined by a convolution of the Lorentzian broadening due to intrawell relaxation and Gaussian broadening due to interwell noise.

In conclusion, we have developed the theory of macroscopic resonant tunneling in flux qubits in the presence of Gaussian low-frequency flux noise. The tunneling rate is given in general by Eq. (13). In the case of strong noise and tunneling between the two lowest energy levels in each well, the resonant peaks have Gaussian shape (8). If the noise source is in equilibrium at temperature T, there is a fundamental relation (10) between the width W of the resonant peaks and shift ϵ_p of their position in energy. The shift ϵ_p provides direct measure of the strength of the quantum component in the flux noise, and leads to splitting of the first resonant peak of flux tunneling. Some of the predictions of the present work has been already tested in experiment [12], which shows, in particular, that Eq. (10) is a convenient tool to determine whether the low-frequency flux noise is produced by an equilibrium quantum source. More experiments, especially with microwave excitations, can be performed to further test our theory.

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