Engineering Correlation and Entanglement Dynamics in Spin Systems

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We show that correlation and entanglement dynamics of spin systems can be precisely controlled and engineered using only a small number of external physical control parameters. We first point out that the correlation dynamics of such systems can be understood in terms of spin-wave propagation, giving a simple physical explanation of the behavior seen in a number of recent works. We then extend this picture to more realistic translationally invariant systems prepared in product states. Since spin waves propagate according to a system's dispersion relation which typically depends on external physical parameters, this insight provides a convenient way to understand how dynamics can be controlled. We demonstrate these ideas in a simple example system, showing that correlations can be made to propagate in well-defined packets whose speed can be engineered in advance, controlled during the evolution, or even stopped altogether.

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Correlations play a predominant role in the study of spin systems. On the one hand, they characterize different phases of matter, and thus can help reveal the mechanisms underlying phase transitions. On the other, they are directly related to the entanglement between different spins, which can be exploited by applications in the field of quantum-information processing. Whereas so far, much of the work on correlations has focused on the static properties of equilibrium systems, an increasing interest in the corresponding dynamical properties has developed over the last few years. The reason is twofold. First, new experimental setups, such as atoms in optical lattices, have reached an unprecedented level of control, allowing physical parameters to be changed during the experiments. Thus theoretical descriptions of the time-dependent properties of such systems have become important. Second, it has been recognized that the way entanglement is created and how it propagates are important fundamental questions in quantum-information theory. In particular, the answers may influence the design of quantum repeaters and networks, whose goal is to establish as much entanglement as possible between different nodes in the shortest possible time.

The time evolution of correlation functions in spin systems has been studied in various scenarios, mainly from a condensed matter physics perspective [1], all of which show correlations propagating at a finite speed. Indeed, Ref. [2] showed that the propagation speed is necessarily finite. On the other hand, information and entanglement propagation in spin systems has mostly been studied from a quantum-information perspective [3]. In contrast with previous work, we will consider to what extent it is possible to *control* the propagation speed and dispersion of the correlations in a translationally invariant system, by manipulating only simple global physical parameters. This may be relevant for the optimal creation of entanglement in spin systems, as well as contributing to a better understanding of how correlations are created in dynamical processes, something that can be tested experimentally in present setups. We will show that, even with this severely limited control, the correlation speed can be engineered while simultaneously keeping dispersion to a minimum, so that correlations can be concentrated between particular spins. Indeed, by manipulating system parameters during the evolution, the speed can be adjusted at will, even to the extent of reducing it to zero, allowing correlations to be frozen at a desired location.

It is instructive to first consider the correlation propagation described in the references given above from a new perspective. In many of those works, correlation propagation can be understood as follows. The spin system is initially prepared in its ground state. A localized, low-energy excitation is then created (e.g., by flipping one spin) and allowed to evolve. Since the low-energy excitations take the form of spin waves, the correlation and entanglement dynamics can be understood as nothing other than propagation of spin waves. The evolution of spin waves is determined by the form of the system's relation, which will typically depend on external physical parameters of the system (e.g., the strength of an external magnetic field). Thus already in these setups, we can manipulate the external parameters to control the dispersion relation, hence control the propagation of correlations. For example, changing the gradient of the dispersion relation will change the propagation speed.

However, the ground state will typically be highly correlated and difficult to prepare, and with the control required to create a local excitation and break the translational symmetry, more sophisticated quantum-repeater setups are possible. Also, it is not clear that the correlations will remain localized; they are likely to disperse rapidly as they propagate. Therefore, we will extend this to systems prepared in translationally invariant, easily created initial states. For example, the fully polarized state with all spins aligned can be prepared by applying a large external magnetic field. As the initial state will be far from the ground state, containing many excitations, the correlation dynamics is the result of

the propagation and interference of a large number of spin waves at many different wave numbers. Nonetheless, we will show that, at least for some simple models, the system can be engineered so that correlations propagate in well-defined localized wave packets with little dispersion. The external parameters can then be used to control the propagation of these correlation packets.

In the following, we will consider a specific model which, despite its simplicity, is sufficiently rich to display most of the features we are interested in, and is simple enough to envisage implementing it experimentally, for instance, using atoms in optical lattices or trapped ions. The XY model for a chain of spin-$\frac{1}{2}$ particles is described by the Hamiltonian $H_{XY} = -\frac{1}{4}\sum_l[(1+\gamma)\sigma_l^x\sigma_{l+1} + (1-\gamma)\sigma_l^y\sigma_{l+1}^y + 2\lambda\sigma_l^z]$, where the σ's are the usual Pauli operators and the sum is over spin indices. The parameter λ can be interpreted as the strength of a global, external magnetic field, whereas γ controls the anisotropy of the interactions. This Hamiltonian can be brought into diagonal form by the well-known procedure [4] of applying Jordan-Wigner, Fourier, and Bogoliubov transformations, giving $H_{XY} = -\frac{i}{4}\sum_k \varepsilon_k(\gamma_k^x\gamma_k^p - \gamma_k^p\gamma_k^x)$ with spectrum $\varepsilon_k = [(\cos(2\pi k/N) - \lambda]^2 + \gamma^2\sin^2[2\pi k/N)]^{1/2}$. The $\gamma_k^{x,p}$ are Majorana operators, related to the more usual Jordan-Wigner fermionic annihilation operator γ_k by $\gamma_k^x = \gamma_k^\dagger + \gamma_k$ and $\gamma_k^p = (\gamma_k^\dagger - \gamma_k)/i$, and they obey canonical anticommutation relations $\{\gamma_k^a, \gamma_l^b\} = 2\delta_{k,l}\delta_{a,b}$.

Ultimately, we are interested in ''connected'' spin-spin correlation functions, for example, the ZZ correlation function $C_{zz}(n,m) = \langle\sigma_n^z\sigma_m^z\rangle - \langle\sigma_n^z\sigma_m^z\rangle$, in which the ''classical'' part of the correlation is subtracted. These are related to the localizable entanglement $L(n,m)$ (the maximum average entanglement between sites n and m extractable by local measurements on all the others: the natural figure of merit for quantum repeaters. In particular, for spin-$\frac{1}{2}$ systems, $L(n,m) \geq C(n,m)$ for any connected correlation function $C(n,m)$ [5]. However, we will start by considering the simpler string correlation functions such as $S_{XX}(i,j) = \langle\sigma_i^x(\prod_{i<k<j}\sigma_k^z)\sigma_j^x\rangle$ [6]). Their behavior will give insight into the more important spin-spin correlations.

Assume the spin chain is initially in some completely separable, uncorrelated state, such as the state with all spins down. The interactions are then switched on and, as this initial state is not an eigenstate of the Hamiltonian (unless $\lambda \to \infty$), the state evolves in time. The initial state is also the vacuum of the Majorana operators $x_l(p_l) = \prod_j\sigma_j^z\sigma_l^{x(y)}$ obtained after applying just the Jordan-Wigner transformation, and it is completely determined by its two-point correlation functions. In other words, the vacuum is a fermionic Gaussian state and can be represented by its covariance matrix $\Gamma_{m,n} = \frac{1}{2}\langle[r_m, r_n]\rangle$, where $r_{2l-1} = x_l$ and $r_{2l} = p_l$.

From the Heisenberg evolution equations, it is simple to show that any evolution governed by a quadratic Hamiltonian corresponds to an orthogonal transformation of the covariance matrix. It is also clear that, as the Fourier and Bogoliubov transformations are canonical (anticommutation-relation-preserving) transformations of the Majorana operators, they similarly leave Gaussian states Gaussian, and they too can be expressed as orthogonal transformations. Thus the time-evolved state of the system is given by a series of orthogonal transformations of the fermionic vacuum:

$$\Gamma(t) = \mathcal{O}\Gamma_{\text{vac}}\mathcal{O}^T, \qquad \mathcal{O} = O_{\text{FT}}^T O_{\text{Bog}}^T O(t). \tag{1}$$

Because of translational invariance, this is a block-Toeplitz matrix, composed of 2×2 blocks G_x at distance x from the main diagonal. In the thermodynamic limit $N \to \infty$ with $\frac{2\pi k}{N} \to \phi$ and $\varepsilon_k \to \varepsilon(\phi) \equiv \varepsilon$,

$$G_x = \int_{-\pi}^{\pi} d\phi \begin{pmatrix} g_0 & g_1 \\ g_{-1} & g_0 \end{pmatrix}, \qquad g_0 = iS\sin(\phi x)\sin(2\varepsilon t)$$
$$g \pm 1 = 2C_\varepsilon S_\varepsilon \sin(\phi x)\sin^2(\varepsilon t) \pm \cos(\phi x)[C^2 + S^2\cos(2\varepsilon t)],$$

where $C_\varepsilon = [\cos(\phi) - \lambda]/\varepsilon(\phi)$, $S_\varepsilon = \gamma\sin(\phi)/\varepsilon(\phi)$, and $x = m - n$. We can now calculate certain string correlations, which are given directly by elements of the covariance matrix. For example, $\langle\sigma_n^x(\prod_{n<i<m}\sigma_i^z)\sigma_m^y\rangle = \frac{1}{i}\Gamma_{2n-1,2m-1} = \sum_{s=\pm 1} s\int_{-\pi}^{\pi} d\phi S\cos(\phi x + 2s\varepsilon t)/2$.

Although the evolution of the string correlations is produced by the collective dynamics of a large number of excitations, this expression has a simple physical interpretation: it is the equation for two wave packets with envelope $S/2$ propagating in opposite directions along the chain, according to a dispersion relation given by the system's spectrum $\varepsilon(\phi)$. This wave-packet interpretation allows us to make quantitative predictions as to how the dynamics will be affected if the system parameters γ and λ are modified. Specifically, modifying the parameters will change the dispersion relation, changing the group velocity of the correlation packets, as well as the rate at which they disperse. (The wave-packet envelopes also depend on the system parameters, so the relevant region of the dispersion relation may also change.) Thus by varying only global physical parameters, we can control the speed at which correlations propagate.

Does this hold true for the more interesting spin-spin correlations? We will show analytically that they have a similar wave-packet description, although in terms of multiple packets propagating simultaneously. This will allow us to predict the behavior of the spin-spin correlation dynamics for different values of the system parameters. In particular, we will show that the correlations can be made to propagate in well-defined packets whose speed can be engineered by tuning the system parameters. Moreover, the propagation speed can be controlled as the system is evolving, so we can speed up or slow down the packets, even to the extent of reducing the speed to zero. We confirm our predictions by numerically evaluating the analytic expressions.

Let us now calculate the spin-spin connected correlation function using the covariance matrix derived above. We have $\sigma_n^z = x_n p_n$, so the ZZ connected correlation function is given by $C_{zz}(x) = \langle x_n p_m \rangle \langle p_n x_m \rangle - \langle x_n x_m \rangle \langle p_n p_m \rangle$, where we have used Wick's theorem to expand the expectation value of the product of four Majorana operators into a sum of expectation values of pairs [7,8]. The latter are given by covariance matrix elements, resulting in the following analytic expression for C_{zz}:

$$C_{zz}(x,t)^2 = \left(\int_{-\pi}^{\pi} d\phi \frac{S}{2} \sum_{s=\pm 1} [s\cos(\phi x - 2s\varepsilon t)]\right)^2 + \left(\int_{-\pi}^{\pi} d\phi CS[\sin(\phi x) - \frac{1}{2}\sum_{s=\pm 1} \sin(\phi x + 2s\varepsilon t)]\right)^2 - \left(\int_{-\pi}^{\pi} d\phi \left[C^2\cos(\phi x) + \frac{S^2}{2}\sum_{s=\pm 1}\cos(\phi x + 2s\varepsilon t)\right]\right)^2. \quad (2)$$

Although more complicated than the string correlations, this expression also describes wave packets evolving according to the same dispersion relation $\varepsilon(\phi)$, albeit multiple packets with different envelopes propagating and interfering simultaneously (three in each direction). In many parameter regimes, broad (in frequency-space) wave packets and a highly nonlinear dispersion relation will cause the correlations to rapidly disperse and disappear. However, we can find regimes in which the wave packets are located in nearly linear regions of the dispersion relation and maintain their shape as they propagate. For example, at $\gamma = 1.1$ and $\lambda = 2.0$, all three wave packets of Eq. (2) are nearly identical and reside in an almost-linear region of the dispersion relation with gradient 2.1, as in Fig. 1 (inset). The spin-spin correlation dynamics will therefore involve well-defined correlation packets propagating at a speed $dx/dt = 2.1$, dispersing only slowly as they propagate. Figure 1 shows the result of numerically evaluating Eq. (2), which clearly confirms the predictions. Note that the magnitude of the correlation function in Fig. 1 is quite small. How tight is this as a lower bound on the localizable entanglement? Numerical simulation using matrix product states coupled with Monte Carlo evaluation of the average over measurements shows that, although the qualitative behavior of the entanglement follows that of the correlations, the localizable concurrence is an order of magnitude larger than this lower bound.

We can engineer a different correlation speed by changing the parameters. E.g., for $\gamma = 10.0$ and $\lambda = 0.9$ we predict from the dispersion relation a higher propagation speed $dx/dt \approx 18$, although at the expense of increased dispersion. Figure 2 shows precisely this behavior.

An even more interesting possibility is controlling the correlation packets as they propagate. If the system parameters are changed continuously in time, the XY Hamiltonian becomes time dependent, and the orthogonal evolution operator $O(t)$ in Eq. (1) is given by a time-ordered exponential $O(t) = T[e^{\int_0^t dt' A(t')}] \equiv \lim_{h\to 0}\prod_{n=1}^{\lfloor t/h \rfloor} e^{A(nh)}$. ($A$ is a time-dependent, antisymmetric matrix determined by the Hamiltonian.) In general, the time ordering is essential. But if the system parameters change slowly in time (i.e., the adiabatic condition is fulfilled), dropping it will give a good approximation to the evolution operator. The state at time t is then just given by evolution under the time average (up to t) of the Hamiltonian. If we remain in a parameter regime for which the relevant region of the dispersion relation is nearly linear, adjusting the parameters changes the gradient without significantly affecting its curvature or the form of the wave packets. Thus, to good approximation, slowly adjusting the parameters should control the speed of the wave packets as they propagate, allowing us to speed them up and slow them down. Numerically evaluating the time-ordered exponential shows this is indeed possible (Fig. 3).

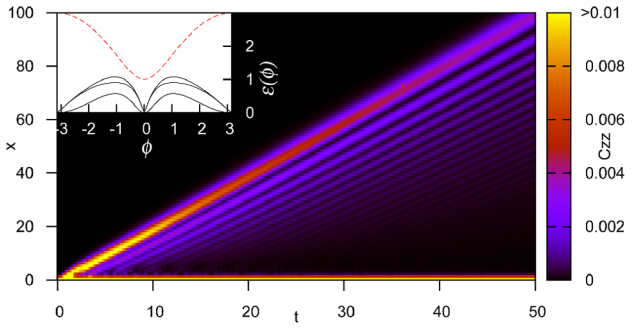

FIG. 1 (color online). For $\gamma = 1.1$, $\lambda = 2.0$, all the wave-packet envelopes ($2S$, $2CS$, and $2S^2$ from Eq. (2), solid curves, inset) are similar in form, peaked around a nearly linear region of the dispersion relation $\varepsilon(\phi)$ (dashed curve, inset) with gradient 2.1. Thus the correlations $C_{zz}(x,t)$ (main plot) propagate in well-defined packets at a speed given by the gradient.

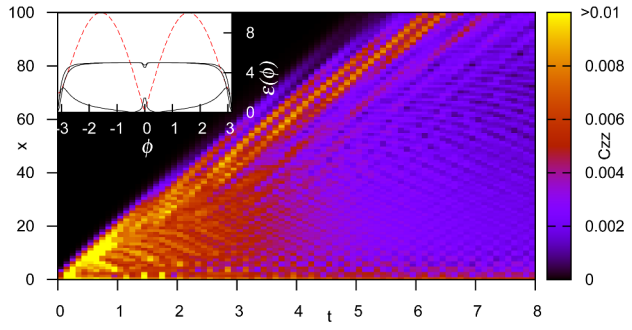

FIG. 2 (color online). For $\gamma = 10.0$, $\lambda = 0.9$, the wave-packet envelopes (solid curves, inset; cf. Fig. 1) are spread over the entire frequency range. The maximum propagation speed of the correlations $C_{zz}(x,t)$ (main plot) is given by the maximum gradient 19.8 of the dispersion relation $\varepsilon\phi$ (dashed curve, inset), faster than in Fig. 1 but with higher dispersion.

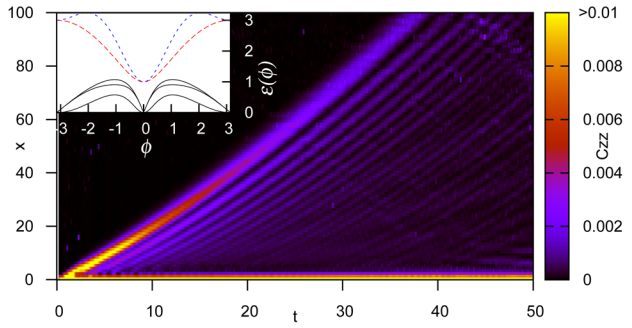

FIG. 3 (color online). Starting from $\gamma = 1.1$, $\lambda = 2.0$ as in Fig. 1, γ and λ are smoothly changed to move from the dashed to the dotted dispersion relation (inset), increasing the correlation speed.

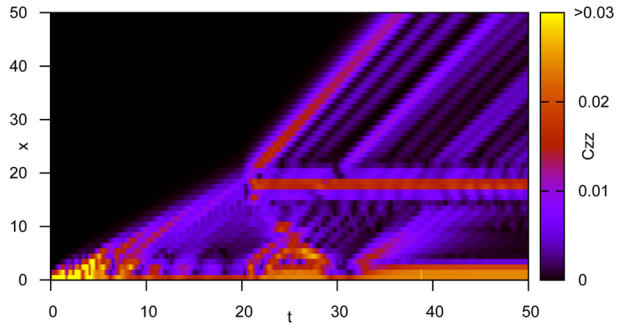

FIG. 4 (color online). The system is initially allowed to evolve with $\gamma_0 = 0.9$ $\lambda_0 = 0.5$, then "quenched" at time $t_1 = 20.0$ to $\gamma_1 = 0.1$, $\lambda_1 = 10.0$. Some of the correlations $C_{zz}(x, t)$ are frozen at the separation ($x \approx 20$) they reached at t_1. Others propagate according to the new dispersion relation, or are "reflected".

Clearly it would be useful to be able to stop the correlations once they reach a desired location. One way would be to simply switch off the interactions. But strictly speaking, this would require more control than is provided by the two parameters defined in the Hamiltonian (there is no value of γ for which all interaction terms vanish), and may be difficult in physical implementations. If the spin model were realized in a solid-state system, for example, switching off the interactions would likely involve fabricating an entirely new system. In any case, we will show that switching off the interactions is not necessary in order to freeze correlations at a specific location.

Instead of changing the parameters continuously, we now consider changing them abruptly. The time-evolved covariance matrix in this scenario can be calculated analytically by the same methods as used above. Suppose the initial system parameters γ_0 and λ_0 are suddenly changed to γ_1 and λ_1 at time t_1. The spin-spin correlations will initially evolve according to Eq. (2), as before. After time t_1, the evolution becomes more complicated. The analogue of Eq. (2) separates into a sum of terms, $C_{zz}(x, t > t_1) = C_{zz}(x, \varepsilon_1 t, t_1) + C_{zz}(x, -\varepsilon_1 t, t_1) + C_{zz}(x, \varepsilon_1 t) + C_{zz}(x, t_1)$, describing wave packets evolving in four different ways: those that initially evolve according to ε_0 and subsequently (for $t > t_1$) evolve according to ε_1 (denoted $C_{zz}(x, \varepsilon_1 t, t_1)$, since they depend on these three variables), those that subsequently evolve according to $-\varepsilon_1$ ($C_{zz}(x, -\varepsilon_1 t, t_1)$), those that only start evolving at t_1 ($C_{zz}(x, \varepsilon_1 t)$), and those that undergo no further evolution after t_1 ($C_{zz}(x, t_1)$), all of which can be seen in Fig. 4. For $t > t_1$, the "frozen" terms are

$$C_{zz}^{t_1}(x, t_1)^2 = \left[\frac{1}{2}\int_{-\pi}^{\pi} d\phi S_{\varepsilon_0} S_{\varepsilon_1} K \sum_{s=\pm 1} \sin(\phi x + 2 s t_1 \varepsilon_0)\right]^2 - \left[\frac{1}{2}\int_{-\pi}^{\pi} d\phi S_{\varepsilon_0} C_{\varepsilon_1} \sum_{s=\pm 1} \cos(\phi x + 2 s t_1 \varepsilon_0)\right]^2,$$

where $K = C_{\varepsilon_0} S_{\varepsilon_1} - C_{\varepsilon_1} S_{\varepsilon_0}$. We can make use of these terms to move correlations to the desired location, then "quench" the system by abruptly changing the parameters, freezing this part of the correlations at that location and leaving the rest to propagate away, as in Fig. 4.

We have shown that controlling correlation propagation in spin systems can often be understood in terms of controlling the spin-wave dispersion relation. We have applied this to a specific example system, showing that by this method even crude global control over the physical parameters of a system can be sufficient to afford precise local control over correlation propagation.

[1] F. Iglói and H. Rieger, Phys. Rev. Lett. **85**, 3233 (2000); L. Amico *et al.*, Phys. Rev. A **69**, 022304 (2004); P. Calabrese and J. Cardy, Phys. Rev. Lett. **96**, 136801 (2006).
[2] S. Bravyi, M. B. Hastings, and F. Verstraete, Phys. Rev. Lett. **97**, 050401 (2006).
[3] T. J. Osborne and N. Linden, Phys. Rev. A **69**, 052315 (2004); S. Bose, Phys. Rev. Lett. **91**, 207901 (2003); M. Christandl *et al.*, *ibid.* **92**, 187902 (2004); P. Karbach and J. Stolze, Phys. Rev. A **72**, 030301(R) (2005); K. Maruyama, T. Iitaka, and F. Nori, *ibid.* **75**, 012325 (2007); J. Eisert *et al.*, Phys. Rev. Lett. **93**, 190402 (2004); S. M. Giampaolo *et al.*, Laser Phys. **16**, 1411 (2006); T. S. Cubitt, F. Verstraete, and J. I. Cirac, Phys. Rev. A **71**, 052308 (2005).
[4] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, England, 2001).
[5] F. Verstraete, M. Popp, and J. I. Cirac, Phys. Rev. Lett. **92**, 027901 (2004); M. Popp, F. Verstraete, M. A. Martin-Delgado, and J. I. Cirac, Phys. Rev. A **71**, 042306 (2005).
[6] I. Affleck *et al.*, Commun. Math. Phys. **115**, 477 (1988).
[7] G. C. Wick, Phys. Rev. **80**, 268 (1950).
[8] T. S. Cubitt and J. I. Cirac (to be published).