Stabilization of Solitons Generated by a Supersonic Flow of Bose-Einstein Condensate Past an Obstacle

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The stability of dark solitons generated by supersonic flow of a Bose-Einstein condensate past an obstacle is investigated. It is shown that in the reference frame attached to the obstacle a transition occurs at some critical value of the flow velocity from absolute instability of dark solitons to their convective instability. This leads to the decay of disturbances of solitons at a fixed distance from the obstacle and the formation of effectively stable dark solitons. This phenomenon explains the surprising stability of the flow picture that has been observed in numerical simulations.

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1. Introduction.—It is well known that plane dark solitons are unstable with respect to transverse (“snake”) perturbations. Such an instability was predicted and studied in [1,2] for the case of shallow solitons described by the Kadomtsev-Petviashvili (KP) equation. The instability of deep dark solitons in a Bose-Einstein condensate (BEC), described by the Gross-Pitaevskii (GP) equation was demonstrated in [3]. These analytical predictions were confirmed by both numerical simulations and experiments with dark solitons in a BEC (see, e.g., [4] and references therein) and with optical solitons (see [5]).

It is worth noting that such soliton instability is not a property of particular equations, but rather a general phenomenon of nondissipative dynamics. Indeed, as follows from [6], the motion of a 1D soliton can be described by the semiclassical Newton equation $m_s d^2X/dt^2 = F_s$, where $m_s = 2dE_s/d(V^2)$ is the “effective mass” of the soliton, E_s is its energy, V its velocity, and F_s is the force acting on the soliton. The crucial point is that in all relevant cases E_s decreases with increasing velocity and hence $m_s < 0$. Let us apply now this equation to an element of a soliton deformed along the y axis. If the wavelength of the perturbation is long enough, the soliton can be considered as a surface with surface tension E_s. Then, according to the Laplace formula, the restoring surface force is equal to $F_s = E_s/R$, where R is the radius of curvature (see [7], Sec. 61). Let the equation of the surface be $X = A\cos(py - \omega t)$. In the linear approximation $R^{-1} \approx d^2X/dy^2 = -p^2X$. This results in an unstable dispersion relation for small oscillations:

$$\omega = \pm i(E_s/|m_s|)^{1/2}p. \quad (1)$$

In the GP equation case for which $E_s \propto (c^2 - V^2)^{3/2}$, where c is the speed of sound, Eq. (1) reduces to the equation $\omega = \pm i[(c^2 - V^2)/3]^{1/2}p$ obtained in [3].

Dark solitons in a uniform BEC are described by the GP equation which we write here in standard dimensionless units with $c = 1$:

$$i\psi_t = -\tfrac{1}{2}\Delta\psi + |\psi|^2\psi, \quad (2)$$

where $\psi(\mathbf{r}, t)$ is the condensate wave function. The solution corresponding to a plane dark soliton moving in the x direction was found in [8] in the form ($k = \sqrt{1 - V^2}$)

$$\psi(x + Vt) = [k\tanh(k(x + Vt)) - iV]\exp(-it). \quad (3)$$

The depth of a dark soliton depends on its velocity V which cannot exceed the sound velocity. The dark soliton is unstable in the whole range of possible parameters. Thus, dark solitons formed by means of density or phase engineering [4,9] are unstable with respect to perturbations depending on the transverse y and z coordinates.

However, there exists another possibility to generate dark solitons in a BEC. As was shown in [10], dark solitons can also be generated by a fast enough flow of a BEC past an obstacle as is illustrated in Fig. 1 where results of a numerical simulation of a two-dimensional flow past a disk-shaped obstacle are shown. We see that the Mach cone [imaginary lines drawn from the origin at angles $\pm\arcsin(1/M)$ with respect to the x axis, $M = 2$ being the supersonic flow velocity at $|x| \to \infty$] separates regions with wave patterns of a different nature. Outside the Mach cone there is a stationary pattern arising due to waves with a Bogoliubov dispersion law radiated by the obstacle. It is possible that these waves have been observed in recent experiments [11,12]. Their theory has been developed in [12–14]. Inside the Mach cone the nonlinear waves are located. In case of large enough obstacles they form dispersive shock waves considered in [15], but for obstacles with the size about a healing length (about unity in our nondimensional units), as in Fig. 1, just one soliton is formed in each symmetrical “shock.” At large enough

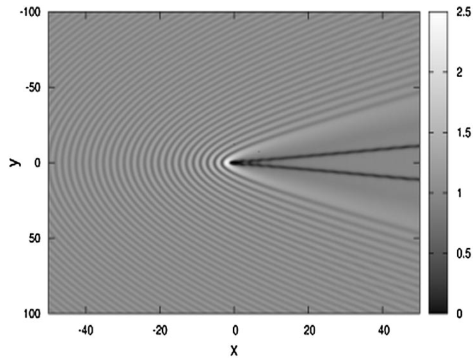

FIG. 1. Wave pattern generated by the flow of a BEC past an obstacle located at the origin of the coordinate system. The flow velocity is directed from left to right and corresponds to the Mach number $M = 2$.

time of evolution and far enough from the obstacle, the profile of a dark soliton tends to a stationary state described by the solution (3) of the GP equation [10]. This means that dark solitons generated by a fast enough flow of a BEC past an obstacle are stable in the reference frame connected with the obstacle contrary to the above mentioned instability of dark solitons in the reference frame where the fluid at infinity is at rest. We will see below that this seeming contradiction is explained by the fact that the instability of solitons in the obstacle reference frame is "convective" only. This means that disturbances are convected from the region at finite distance behind the obstacle and, hence, the solitons there become stable.

It is also known that at small enough velocity of the flow the situation is different. Numerical solutions of the GP equation demonstrate that a BEC flow past a cylindrical obstacle starts to generate vortices at subsonic velocity $M \cong 0.45$ (see [16–18]). However, the frequency of vortex generation increases with increasing flow velocity. Therefore one can expect that at large enough velocity vortices are generated so often that the mean distance between them becomes less than their radial size so that their separation from each other takes a long time which results in the formation of dark oblique solitons at a finite distance from the obstacle.

2. Convective and absolute instabilities.—The presence of a positive imaginary part in the dispersion law $\omega = \omega(p)$ of the soliton oscillations does not mean in itself that an arbitrary perturbation in a given point will grow with time. The stability of the soliton is determined by the asymptotic behavior of wave packets built from harmonic waves. Let the Fourier transform of the initial density perturbation be δn_p. Then the time dependence of the perturbation is defined by the equation

$$\delta n(t, y) \propto \int_{-\infty}^{\infty} \delta n_p e^{i[py-\omega(p)t]} dp. \quad (4)$$

Hence, there are two possibilities. In one case, the perturbation increases without limit at any fixed value of y. This situation corresponds to absolute instability. In the other case the packet is carried away by the flow along the soliton plane so fast that the perturbation tends to zero as $t \to \infty$ at any fixed value of y. This corresponds to convective instability. We emphasize that from a practical point of view a convective instability does not prevent the observation of a finite size soliton in experiments and does not violate the stability of numerical simulations.

The asymptotic behavior of the perturbation (4) at $t \to \infty$ can be investigated by deformation of the contour of integration in the complex p plane. The criterion distinguishing absolute and convective instabilities can be formulated as follows (see [19], Chap. VI): (i) in the complex p plane the function $p(\omega)$ has values lying in upper and lower half-planes for $\mathrm{Im}\omega \equiv \omega'' \to +\infty$; (ii) when we decrease ω'' then for some values of ω the two values of p moving from opposite half-planes coincide in the branching point p_{br} of the function $p(\omega)$; (iii) if these branching points correspond to values of ω_{br} lying in the upper ω complex half-plane, then we have absolute instability; otherwise the instability is convective. The asymptotic behavior of the perturbation at $t \to \infty$ is $\delta n(t, y) \propto e^{i(p_{\mathrm{br}}y-\omega_{\mathrm{br}}t)}/\sqrt{t}$, where $p_{\mathrm{br}} = p(\omega_{\mathrm{br}})$.

This investigation demands knowledge of the dispersion law $\omega(p)$ for all values p so that knowledge of just the long wavelength approximation (1) is not sufficient for this aim. Before going to concrete calculation, we have to make the following general remark. The solution (3) describes a soliton in the absence of a flow along its plane. In our problem the soliton is created by the flow moving with respect to the obstacle with velocity $M(= Mc)$. Let θ be the angle between a vector normal to the soliton plane and the direction of the flow. Then the normal component of velocity is $V = M\cos\theta$ and the flow has a component $u = M\sin\theta$ along the soliton plane. [A solution of (2) corresponding to this "oblique" soliton has been obtained in [10].] Since we want to investigate the stability of the soliton in the "laboratory" coordinate frame with the obstacle at rest, we have to transform the dispersion law $\omega_0(p)$ of the oscillation of a "direct" soliton (3) to the laboratory reference frame which means that $\omega_0(p)$ should be replaced by $\omega(p) = up + \omega_0(p)$.

3. Shallow dark solitons.—The investigation of the nature of instability of the soliton (3) is a quite complicated problem. As a first step, we will study a soliton with a small amplitude, which moves with the near-sonic velocity V, $1 - V \ll 1$. In this case the dispersion law can be found analytically. As is known, in the small amplitude limit the GP equation can be reduced to the KP equation (see, e.g., [3])

$$(\tilde{n}_t - \tilde{n}_x - \tfrac{3}{2}\tilde{n}\tilde{n}_x + \tfrac{1}{8}\tilde{n}_{xxx})_x = \tfrac{1}{2}\tilde{n}_{yy}, \quad (5)$$

where $n = 1 + \tilde{n}$ and we used the same notations t, x, y for the new variables.

This equation has the soliton solution

$$\tilde{n}(x, t) = -s/\cosh^2[\sqrt{s}(x + Vt)], \tag{6}$$

where $s = 2(1 - V)$. Since in the small amplitude limit we have $1 - V^2 \cong 2(1 - V)$, this solution is an approximation to the exact solution (3). Small oscillations of the soliton (6) were studied in [2,20] where, for waves propagating with wave number p along soliton, the spectrum $\omega_0(p) = i\Gamma(p)$ was obtained, which after transformation to the "obstacle" frame takes the form

$$\omega = \omega(p) = up + i\Gamma(p) = up + i(p/\sqrt{3})\sqrt{s - 2p/\sqrt{3}}. \tag{7}$$

The above criterion can be easily applied to the dispersion relation (7). Indeed, at $\omega'' \to +\infty$ this relation yields three values of p one of which $[p \propto -(\omega'')^{3/2}]$ lies on the real axis, and the other two $[p \propto (\omega'')^{3/2} e^{\pm \pi i/3}]$ are located in the upper and lower p half-planes. Then, in the branching point we have $d\omega(p)/dp = 0$ and this equation gives after a simple calculation the critical value of p:

$$p_{\rm br} = [s + u^2 - u\sqrt{u^2 - s}]/\sqrt{3}. \tag{8}$$

Substitution of this value into Eq. (7) yields the critical value of ω:

$$\omega_{\rm br} = p_{\rm br}(\sqrt{u^2 - s} + 2u)/3. \tag{9}$$

For $u^2 < s$ we get $\omega''_{\rm br} = [(s - u^2)/3]^{3/2} > 0$ which means the instability is absolute. However, $p_{\rm br}$ and $\omega_{\rm br}$ have real values for $u^2 > s$ and in this case the instability is convective. Thus, the transition from absolute to convective instability takes place at $u^2 > s$, that is at $M^2 > 1$. The shallow oblique solitons are convectively unstable therefore for supersonic flow of a BEC past an obstacle.

4. Convective instability criterion for deep dark solitons.—In the case of deep dark solitons there is no analytical expression for the full instability spectrum, although its asymptotic expressions are known: for small $p \ll q$ we have (see [3])

$$\Gamma^2(p) = k^2p^2/3 - (2 - k^2)p^3/(3\sqrt{3}), \qquad p \ll q, \tag{10}$$

where $k^2 = 1 - V^2$ and

$$q = [-(1 + V^2) + 2\sqrt{V^4 - V^2 + 1}]^{1/2}. \tag{11}$$

Notice that in the limit $p \to 0$ Eq. (11) coincides with the general result (1) obtained for the GP equation. For $q - p \ll q$ we have (see [21])

$$\Gamma^2(p) = q(q - p)/\beta(k), \qquad q - p \ll q, \tag{12}$$

where $\beta(k)$ is evaluated numerically. We approximate this behavior by an interpolating function

$$\Gamma^2(p) = f(p)(q - p) \tag{13}$$

with $f(p) = a(k)p^2 + b(k)p^3 + c(k)p^4$, where

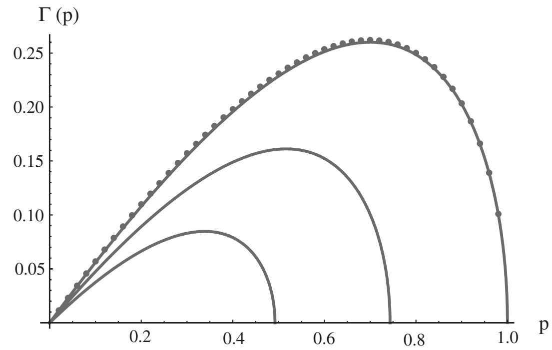

FIG. 2. Growth rate approximated by Eq. (13) with $f(p)$ given by the interpolation formula as a function of wave vector p of a harmonic wave propagating along the oblique dark solitons with $V = 0.7$, $V = 0.5$, and $V = 0$. The exact dependence found numerically for $V = 0$ is shown by dots for illustration.

$$a(k) = k^2/(3q), \qquad b(k) = [k^2/q - (2 - k^2)/\sqrt{3}]/(3q),$$

$$c(k) = [3/\beta(k) - 2k^2 + q(2 - k^2)/\sqrt{3}]/(3q^3),$$

and $\beta(k)$ can also be approximated as $\beta(k) = 3/[(1 + \sigma k^2)k^2]$, $\sigma \cong 0.596$. This interpolating expression yields the dependence of Γ on p shown in Fig. 2 for several values of V.

On the contrary to the shallow soliton case, an expression for the critical flow velocity $M_{\rm cr}$ cannot be given in explicit form. Therefore we shall follow Sturrock's [22] formulation of the criterion and notice that the function $p = p(\omega)$ determined implicitly by the equation $\omega(p) = up + i\Gamma(p, V)$, changes its behavior at $u = u_{\rm cr}$: for $u < u_{\rm cr}$ it is represented in the complex p plane by disconnected curves, whereas for $u > u_{\rm cr}$ these curves are connected with each other. Therefore a "spacelike" Fourier representation of a wave packet of disturbance cannot be transformed to its "timelike" representation for $u < u_{\rm cr}$ and in this case the instability is absolute. For $u > u_{\rm cr}$ we can deform the contour p so that a single-valued function $\omega(p)$ is defined for all $p > 0$ and hence a spacelike packet is also a timelike packet which means that the instability is convective (see [22] for details). At the branching point of $p = p(\omega)$ we have $d\omega/dp = 0$, that is the values $p_{\rm br}$ satisfy the equation

$$u = -id\Gamma/dp \equiv -i\Gamma'(p, V), \tag{14}$$

where $\Gamma(p, V)$ has either real or purely imaginary values for real p. Hence, the critical value of u corresponds to the appearance of a double zero $p_{\rm br}$ of Eq. (14) on the real axis of p as it takes place in the above considered case of shallow solitons, that is $dp_{\rm br}/du = \infty$ at $u = u_{\rm cr}$. (In other words, $p_{\rm br}$ has here a branching point as a function of u.) Differentiation of Eq. (14) with respect to u then gives the equation $d^2\Gamma/dp^2|_{p=p_{\rm cr}} = 0$ for the corresponding critical value $p_{\rm cr}(V)$. Substitution of $p_{\rm cr}$, found in this way, into squared Eq. (14) yields the equation

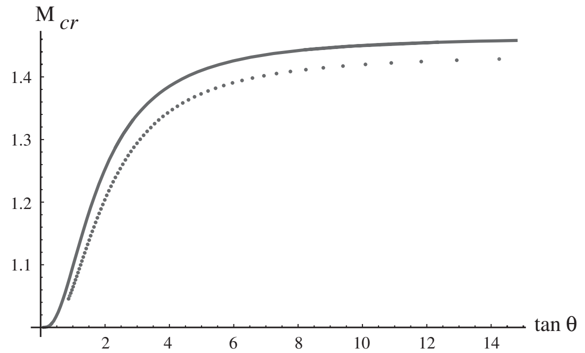

FIG. 3. The boundary between regions of convective and absolute instabilities in the plane of parameters ($\tan\theta$, $M_{\rm cr}$). Below this curve the instability is absolute and above it it is convective. Dots correspond to the transition points calculated from a numerical solution of the eigenvalue problem for the instability spectrum.

$$M_{\rm cr}^2 \sin^2\theta = -[\Gamma'(p_{\rm cr}(M_{\rm cr}\cos\theta), M_{\rm cr}\cos\theta)]^2, \tag{15}$$

which determines implicitly the critical value $M_{\rm cr}$ of the transition from absolute instability to a convective one as a function of the angle θ. For $\Gamma(p)$ given by Eq. (7) these formulas reproduce the above results derived for shallow solitons. The plot of $M_{\rm cr}(\tan\theta)$ calculated for the instability spectrum defined by the interpolation formula (13) is shown in Fig. 3 by a solid line. As we see, the curve obtained with the use of the interpolation formula agrees well enough with the results derived from the numerically calculated instability spectrum.

In experiment [11], the obstacle was created by a laser beam. Hence the flow of BEC past a cylindrical obstacle was actually studied. Our results can be applied to this case without any changes due to the isotropy of the growth rate spectrum $\Gamma(p)$, $p = (p_x^2 + p_y^2)^{1/2}$, in momentum space. Because of this isotropy, the unstable modes grow up with the same increments in all directions and if the addition of convection can remove a wave packet built from unstable modes from a region finite in the x direction, then it removes it from a region finite in the y direction, too.

5. Conclusion.—Thus, for $M < 1$ oblique solitons are absolutely unstable with respect to the transverse instability leading to their decay to vortex pairs of opposite polarities. However, for supersonic flow $M > 1$ a region of angles θ appears where oblique solitons are only convectively unstable which provides the possibility of formation of oblique solitons generated by the flow of a BEC past an obstacle. For M greater than some maximal value $M_{\max} \cong 1.46$, oblique solitons are only convectively unstable for any angle θ inside the Mach cone. These results explain the stability of oblique solitons observed in numerical simulations [10]. We suppose that this method of generation of dark solitons in a BEC presents new possibilities for their experimental study.

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