K_{l3} Semileptonic Form Factor from $(2 + 1)$-Flavor Lattice QCD

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We present the first results for the K_{l3} form factor from simulations with $2 + 1$ flavors of dynamical domain wall quarks. Combining our result, namely, $f_+(0) = 0.964(5)$ with the latest experimental results for K_{l3} decays leads to $|V_{us}| = 0.2249(14)$, reducing the uncertaintity in this important parameter. For the $O(p^6)$ term in the chiral expansion we obtain $\Delta f = -0.013(5)$.

The increasing precision with which the unitarity of the CKM matrix [1] can be tested is an important tool for exploring the limits of the standard model. One such unitarity relation is

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1, \qquad (1)$$

whose uncertainty is dominated by the precision of $|V_{us}|$. In order to obtain $|V_{us}|$ from experimental measurements of the rate for an $s \to u$ decay process, it is necessary to quantify the corresponding nonperturbative QCD effects. In this Letter we present first lattice results from simulations with $2 + 1$ flavors of domain wall quarks, which respect chiral and flavor symmetries to high accuracy, for the evaluation of the form factor $f_+(0)$ necessary to determine $|V_{us}|$ from $K \to \pi\ell\nu_\ell$ ($K_{\ell3}$) semileptonic decays. Precise knowledge of $f_+(0)$ is crucial also for deducing $|V_{td}|$ from a measurement of $K \to \pi^0\nu\bar\nu$.

Our determination of $f_+(0)$ includes estimates of all systematic errors (chiral and q^2 extrapolations, discretization, and finite-volume effects), and reduces the combined theoretical and experimental error in $|V_{us}|$ from the PDG(2006) result of 0.2257(21) to [2]

$$|V_{us}| = 0.2249(14). \qquad (2)$$

The combination $|V_{us}f_+(0)|$ can be obtained from the experimental rate for $K_{\ell3}$ decays

$$\Gamma_{K \to \pi l\nu} = C_K^2 \frac{G_F^2 m_K^5}{192\pi^3} I S_{EW}[1 + 2\Delta_{SU(2)}$$
$$+ 2\Delta_{EM}]|V_{us}|^2|f_+(0)|^2, \qquad (3)$$

where I is the phase space integral which can be evaluated from the shape of the experimental form factor, and $\Delta_{SU(2)}$, S_{EW}, Δ_{EM} contain the isospin breaking, short distance electroweak and long distance electromagnetic corrections, respectively. $f_+(0)$ is the form factor defined from

the $K \to \pi$ matrix element of the weak vector current, $V_\mu = \bar{s}\gamma_\mu u$, evaluated at zero momentum transfer

$$\langle\pi(p')|V_\mu|K(p)\rangle = (p_\mu + p'_\mu)f_+(q^2)$$
$$+ (p_\mu - p'_\mu)f_-(q^2), \qquad (4)$$

where $q^2 = (p - p')^2$. PDG(2006) quotes [4,5]

$$|V_{us}f_+(0)| = 0.2169(9), \qquad (5)$$

hence, in order to obtain $|V_{us}|$ at a precision commensurate with current experiments, we need to determine $f_+(0)$ with an error of less than 1%.

In chiral perturbation theory (ChPT), $f_+(0)$ is expanded in terms of the light pseudoscalar meson masses

$$f_+(0) = 1 + f_2 + f_4 + \ldots, \qquad (f_n = \mathcal{O}(m_{\pi,K,\eta}^n)). \quad (6)$$

Current conservation ensures that in the $SU(3)_{\text{flavor}}$ limit $f_+(0) = 1$, hence f_2 and f_4 are *small*. Additionally, as a result of the Ademollo-Gatto Theorem [7], which states that f_2 receives no contribution from local operators appearing in the effective theory, f_2 is determined unambiguously in terms of m_π, m_K and f_π, and takes the value $f_2 = -0.023$ at the physical values of the meson masses [8]. Our task is now reduced to one of finding

$$\Delta f = f_+(0) - (1 + f_2). \qquad (7)$$

Until recently, the canonical estimate of $\Delta f = -0.016(8)$ was due to Leutwyler and Roos (LR) [8], whereas more recent ChPT based phenomenological analyses favor a value consistent with zero (see Table I). These determinations, however, require model input; the 50% error in the LR result, for example, was estimated within the context of a simple quark model. Hence a model independent determination of Δf with a reliable error estimate is necessary. We compile recent lattice and phenomenological results in Table I. Our lattice calculation has been discussed in

TABLE I. Summary of ChPT and lattice results.

Ref.	$f_+(0)$	Δf	m_π [GeV]	a [fm]	N_f
[8]	0.961(8)	−0.016(8)			
[9]	0.978(10)	+0.001(10)			
[10]	0.984(12)	+0.007(12)			
[11]	0.974(11)	−0.003(11)			
[12]	0.960(5)(7)	−0.017(5)(7)	≳ 0.5	0.07	0
[13]	0.968(9)(6)	−0.009(9)(6)	≳ 0.49	0.12	2
[14][a,b]	0.962(6)(9)	−0.015(6)(9)	c	c	2 + 1
[15][a]	0.967(6)	−0.010(6)	≳ 0.55	0.09	2
[16][a]	0.965(2)	−0.012(2)	≳ 0.5	0.08	2
This work	0.964(5)	−0.013(5)	≳ 0.33	0.114	2 + 1

[a]Results in conference proceedings only.
[b]Used slope of experimental form factor as input.
[c]Information not provided.

preliminary form in [17] and we now finalize our results with the inclusion of the complete set of data and a careful estimate of all systematic errors.

We simulate with $N_f = 2 + 1$ dynamical flavors generated with the Iwasaki gauge action [18] at $\beta = 2.13$, which corresponds to an inverse lattice spacing $a^{-1} = 1.73(3)$ GeV ($a = 0.114(2)$ fm) [19,20], and the domain wall fermion action [21] with a residual mass of $am_{res} = 0.00315(2)$ [19,20]. The simulated strange quark mass, $am_s = 0.04$, is close to its physical value [20], and we choose four values for the light quark masses, am_{ud}, which correspond to pion masses as light as 329 MeV [19,20]. The calculations are performed on two volumes, 16^3 (($1.83)^3$ fm^3) and 24^3 (($2.74)^3$ fm^3), at each quark mass, except the lightest mass which is only simulated on the larger volume. Simulation details are summarized in Table II and more details can be found in [19,20].

We start by rewriting the vector form factors given in (4) to define the scalar form factor

$$f_0(q^2) = f_+(q^2) + \frac{q^2}{m_K^2 - m_\pi^2} f_-(q^2), \qquad (8)$$

which can be obtained on the lattice at $q_{max}^2 = (m_K - m_\pi)^2$ with high statistical accuracy [12,22]. In Table III we present our results for $f_0(q_{max}^2)$ for each of the simulated quark masses and volumes.

For each quark mass, in addition to evaluating $f_0(q^2)$ at $q^2 = q_{max}^2$, we determine the form factor at several nega-

tive values of q^2, allowing us to interpolate to $q^2 = 0$. Specifically, in the notation of (4), we evaluate the form factor with $|\vec{p}'| = 0$, $|\vec{p}| = p_L$ or $|\vec{p}| = \sqrt{2}p_L$ where $p_L = 2\pi/L$ and L is the spatial extent of the lattice, and also with $|\vec{p}| = 0$, $|\vec{p}'| = p_L$ or $|\vec{p}'| = \sqrt{2}p_L$. To obtain $f_0(q^2)$ we use standard ratio techniques [12,13,22], which do not require normalization of the vector current.

In order to gain the maximum amount of information from limited data, we perform a simultaneous fit to both the q^2 and quark mass dependencies using the ansatz

$$f_0(q^2, m_\pi^2, m_K^2) = \frac{1 + f_2 + (m_K^2 - m_\pi^2)^2(A_0 + A_1(m_K^2 + m_\pi^2))}{1 - q^2/(M_0 + M_1(m_K^2 + m_\pi^2))^2}, \qquad (9)$$

with four fit parameters A_0, A_1, M_0, M_1, and where f_2 is the NLO term appearing in the chiral expansion (6), evaluated by inserting the lattice results for m_π, m_K and the physical value for f_π (132 MeV) into the expression appearing in ChPT [8] at each quark mass [23].

The expression (9) is well motivated since we know from the Ademollo-Gatto Theorem that to leading order $\Delta f \propto (m_s - m_{ud})^2$, hence we expect

$$f_0(0) = 1 + f_2 + (m_K^2 - m_\pi^2)^2(A_0 + A_1(m_K^2 + m_\pi^2)), \quad (10)$$

which incorporates the correct $SU(3)_{flavor}$ limit, $f_+(0) = 1$, to be a good phenomenological ansatz for the mass dependence of $f_0(0) = f_+(0)$. This motivates the numerator in (9), while the denominator comes from simply including a quark mass dependence into the standard pole dominance form

$$f_0(q^2) = f_0(0)/(1 - q^2/M^2), \qquad (11)$$

where M is a pole mass, which has been shown to describe the q^2 dependence of lattice results of $f_0(q^2)$ very well [12,13].

The traditional approach of sequentially interpolating in q^2 (11) followed by chiral extrapolation of $f_+(0)$ (10) should agree with our simultaneous fit (9). Fitting the 24^3 data only yields excellent agreement (shown in the final two rows of Table IV), with a reduced error evident in the simultaneous fit, which we therefore take as our best result. For the 16^3 data the pole fits generally have a poor $\chi^2/$dof. We also find that the simultaneous and sequential fits to the q^2 and mass dependence for the 16^3 data differ at 1.2σ. Consequently, we only use the 16^3 data to check that the

TABLE II. Simulation parameters: bare light quark mass (am_{ud}), pion (m_π) and kaon (m_K) masses for both volumes.

	$16^3 \times 32$		$24^3 \times 64$	
am_{ud}	m_π [GeV]	m_K [GeV]	m_π [GeV]	m_K [GeV]
0.03	0.674(11)	0.723(12)	0.671(11)	0.719(12)
0.02	0.557(9)	0.666(11)	0.556(9)	0.663(11)
0.01	0.428(7)	0.614(10)	0.416(7)	0.604(10)
0.005	⋯	⋯	0.329(5)	0.575(9)

TABLE III. Results for $f_0(q_{max}^2)$, where $q_{max}^2 = (m_K - m_\pi)^2$.

	$16^3 \times 32$		$24^3 \times 64$	
am_{ud}	q_{max}^2 [GeV^2]	$f_0(q_{max}^2)$	q_{max}^2 [GeV^2]	$f_0(q_{max}^2)$
0.03	0.002 33(4)	1.000 35(3)	0.002 35(4)	1.000 29(6)
0.02	0.011 78(24)	1.002 41(19)	0.011 52(20)	1.001 92(34)
0.01	0.034 75(66)	1.014 36(81)	0.035 24(62)	1.008 87(89)
0.005	⋯	⋯	0.060 70(107)	1.021 43(132)

TABLE IV.   Results for $f_+(0)$ using pole dominance (11) and quadratic (12) fits to each data set, together with the chiral extrapolations using (10) with the $24^3 \times 64$ data only. The final row gives the results for simultaneous q^2 and quark mass fits [(9) and (13)] using the same data sets.

	Pole		Quadratic	
am_{ud}	$f_+(0)$	χ^2/dof	$f_+(0)$	χ^2/dof
$16^3 \times 32$				
0.03	0.99925(8)	5.0/3	0.99938(12)	4.2/2
0.02	0.9951(6)	13.5/3	0.9959(9)	13.0/2
0.01	0.9889(26)	13.9/3	0.9866(33)	10.9/2
$24^3 \times 64$				
0.03	0.9991(2)	2.1/3	0.9990(2)	1.5/2
0.02	0.9960(7)	2.3/3	0.9962(9)	1.9/2
0.01	0.9841(29)	10.4/3	0.9806(39)	7.7/2
0.005	0.9774(35)	4.0/3	0.9749(59)	2.7/2
Chiral	0.9644(39)	3.4/2	0.9622(61)	5.1/2
Sim. fit	0.9644(33)	28.7/16	0.9610(43)	26.4/14

finite-volume effects are small. Tables III and IV demonstrate that this is the case.

We present the results from a fit to the $24^3 \times 64$ data sets using (9) in Fig. 1. Here the curve shows the fit function at the physical meson masses, while the difference $f_0(q^2, m_\pi^{\mathrm{latt}}, m_K^{\mathrm{latt}}) - f_0(q^2, m_\pi^{\mathrm{phys}}, m_K^{\mathrm{phys}})$ has been subtracted from our raw data points and the small scatter is indicative of the quality of our fit.

The quark mass dependence of (9) is presented in Fig. 2. The solid line represents the fit function evaluated at $q^2 = 0$, plotted as a function of m_π^2, while the dashed line is the contribution coming from the $\mathcal{O}(p^4)$ terms in the chiral expansion, $1 + f_2$. Our results clearly indicate a sizeable, negative value for $\Delta f = -0.013(3)$, in contrast to the recent ChPT based results of [9–11]. In Fig. 2 we also overlay the results given in Table IV for $f_0(0)$ obtained from individual pole fits on each of our ensembles and earlier $N_f = 2$ results [13].

So far, we have assumed a pole dominance behavior (9) in our lattice data, whose q^2 dependence differs marginally
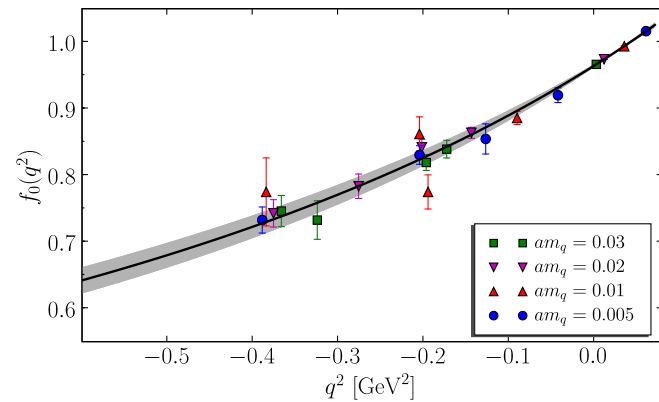
at NLO from an expression obtained in ChPT [9]. In order to estimate the systematic error due to this choice, we also present in Table IV results for fits to our data using a quadratic ansatz

$$f_0(q^2) = a_0 + a_1 q^2 + a_2 q^4, \qquad (12)$$

together with a chiral extrapolation using (10). A simultaneous fit similar to (9) is possible via

$$\begin{aligned} f_0(q^2, m_\pi^2, m_K^2) = {} & 1 + f_2 + (m_K^2 - m_\pi^2)^2 \\ & \times (A_0 + A_1 + A_2(m_K^2 + m_\pi^2)) \\ & + (A_3 + (2A_0 + A_1)(m_K^2 + m_\pi^2))q^2 \\ & + (A_4 - A_0 + A_5(m_K^2 + m_\pi^2))q^4. \quad (13) \end{aligned}$$

The form of this ansatz is motivated by the expression obtained in ChPT [9]. We quote the result from a fit to the $24^3 \times 64$ data using (13) in the last row of Table IV, where we find that the results of the two fits, (9) and (13), agree within statistical precision and we take the difference (0.0034) as an estimate of the systematic error in choosing (9) as our preferred ansatz.

Recently, an alternative parametrization, obtained by using analyticity and crossing symmetry, has been proposed [24]. We find that fitting our data using this ansatz leads to results that lie within the systematic uncertainty of 0.0034 discussed above.

Our simulations are performed with a strange quark mass ($am_s + am_{\mathrm{res}} \simeq 0.043$) which is heavier than the physical mass ($am_s + am_{\mathrm{res}} \simeq 0.037$). Both (9) and (13) are modeled according to ChPT and this mass difference is corrected when we insert the physical kaon mass to obtain our final result. This correction is accurate in as much as our extrapolation model describes our data, and any error introduced is included in our estimate of the systematic error. Future simulations will include a second valence strange quark mass to decrease the reliance on our fit model.
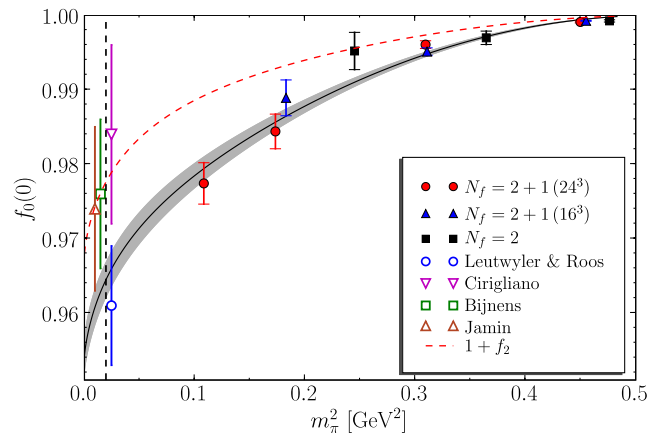

FIG. 1 (color online).   Scalar form factor, $f_0(q^2)$, together with the simultaneous fit of (9) as described in the text.

FIG. 2 (color online).   Scalar form factor, $f_0(0)$, together with the simultaneous fit (solid line) on the 24^3 data (red circles) using (9).
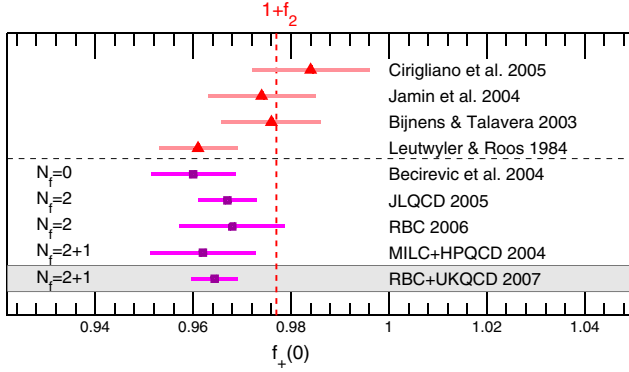
FIG. 3 (color online).   Comparision with other determinations of $f_+(0)$.

Finally, since we simulate at a single lattice spacing, we are unable to extrapolate to the continuum limit. However, leading lattice artefacts with domain wall fermions are of $O(a^2 \Lambda_{QCD}^2)$; assuming $\Lambda_{QCD} \sim 300$ MeV we estimate these to be no larger than $\approx 4\%$ (of $1 - f_+$). A comparison of the pion and kaon decay constants obtained from our simulations with their physical values provides a test for the reliablity of our result. After including the effects to NLO due to chiral logs, we find f_π and f_K about 4% below experiment [20], which is consistent with our estimated scaling error. We will explicitly check this for $K_{\ell 3}$ decays on our new ensemble which is being generated on a finer lattice. Note that our current uncertainty is dominated by statistics and the chiral and q^2 extrapolations and not by the discretization error. Hence our final result is

$$f_+(0) = 0.9644(33)(34)(14), \qquad (14)$$

where the first error is statistical, and the second and third are estimates of the systematic errors due to our choice of parametrization (9) and lattice artefacts, respectively. To put this result in context, we compare our value with other determinations of $f_+(0)$ in Fig. 3. We see that our result agrees very well with the Leutwyler-Roos value [8] and earlier lattice calculations [12–15]. In particular, we note that our findings prefer a sizeable, negative value for $\Delta f = -0.0129(33)(34)(14)$, in contrast to recent ChPT based phenomenological results [9–11].

Using $|V_{us} f_+(0)| = 0.2169(9)$ from PDG(2006) [4,25]

$$|V_{us}| = 0.2249(9)_{\text{exp}}(11)_{f_+(0)}, \qquad (15)$$

and combined with $|V_{ud}| = 0.973\,77(27)$ [4] we find

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 - \delta, \quad \delta = 0.0012(8), \quad (16)$$

compared with the PDG(2006) [4] result, $\delta = 0.0008(10)$. Further reduction in the lattice error is imperative. Our q^2 interpolation systematic is removable in principle [26] and we are in the process of addressing both this and discretization systematics with a new set of simulations.

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