Unitary Parametrization of Perturbations to Tribimaximal Neutrino Mixing

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Current experimental data on neutrino mixing are very well described by tribimaximal mixing. Accordingly, any phenomenological parametrization of the Maki-Nakagawa-Sakata-Pontecorvo matrix must build upon tribimaximal mixing. We propose one particularly natural parametrization, which we call "triminimal." The three small deviations of the Particle Data Group angles from their tribimaximal values, and the PDG phase, parametrize the triminimal mixing matrix. As an important example of the utility of this new parametrization, we present the simple resulting expressions for the flavor-mixing probabilities of atmospheric and astrophysical neutrinos. As no foreseeable experiment will be sensitive to more than second order in the small parameters, we expand these flavor probabilities to second order.

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Neutrino physics has entered the precision era [1]. In the next decade, the uncertainty in our knowledge of neutrino masses and mixing angles will decrease considerably. Many of the proposed models of neutrino mass and mixing will be tested. The Maki-Nakagawa-Sakata-Pontecorvo (MNSP) neutrino mixing matrix describes the unitary transformation between the mass and flavor bases of the neutrinos. In vacuum, it is given by $U_{\alpha j} = \langle \alpha | j \rangle$, with $\alpha = e, \mu, \tau$ and $j = 1, 2, 3$; i.e., rows are labeled from top to bottom by the flavor indices, and columns are labeled left to right by mass-eigenstate indices.

Three mixing angles and a phase comprise the conventional parametrization of the vacuum mixing matrix, as established by the Particle Data Group (PDG) [2]:

$$U = R_{32}(\theta_{32}) U_\delta^\dagger R_{13}(\theta_{13}) U_\delta R_{21}(\theta_{21}) = \begin{pmatrix} c_{21}c_{13} & s_{21}c_{13} & s_{13}e^{-i\delta} \\ -s_{21}c_{32} - c_{21}s_{32}s_{13}e^{i\delta} & c_{21}c_{32} - s_{21}s_{32}s_{13}e^{i\delta} & s_{32}c_{13} \\ s_{21}s_{32} - c_{21}c_{32}s_{13}e^{i\delta} & -c_{21}s_{32} - s_{21}c_{32}s_{13}e^{i\delta} & c_{32}c_{13} \end{pmatrix}, \tag{1}$$

where $R_{jk}(\theta_{jk})$ describes a rotation in the jk-plane through angle θ_{jk}, $U_\delta = \mathrm{diag}(e^{i\delta/2}, 1, e^{-i\delta/2})$, and $s_{jk} = \sin\theta_{jk}$, $c_{jk} = \cos\theta_{jk}$. We have omitted two additional Majorana phases, as they do not affect neutrino oscillations.

From Eq. (1), one gleans that the PDG mixing angles are related to certain observable moduli of matrix elements as

$$\sin^2\theta_{13} = |U_{e3}|^2, \tag{2}$$

$$\sin^2\theta_{21} = |U_{e2}|^2/(1 - |U_{e3}|^2), \tag{3}$$

$$\sin^2\theta_{32} = |U_{\mu 3}|^2/(1 - |U_{e3}|^2). \tag{4}$$

In the order listed, these important moduli are inferred from terrestrial (long-baseline or reactor) data, solar data, and atmospheric data. Finally, the CP-invariant of Jarlskog is given by

$$J_{CP} = -\mathrm{Im}\{U_{e1}U_{\mu 3}U_{e3}^* U_{\mu 1}^*\}. \tag{5}$$

In terms of the PDG parametrization in Eq. (1), this is $J_{CP} = \frac{1}{8}\sin 2\theta_{21}\sin 2\theta_{32}\sin 2\theta_{13}\cos\theta_{13}\sin\delta$. Global 3-flavor fits to data give the following (1σ) and 3σ ranges for the PDG mixing angles [3]:

$$\sin^2\theta_{21} = 0.32(\pm 0.02)^{+0.08}_{-0.06},$$
$$\sin^2\theta_{32} = 0.45(\pm 0.07)^{+0.20}_{-0.13}, \qquad \sin^2\theta_{13} < (0.02)0.050. \tag{6}$$

The central values of these inferred mixing angles are quite consistent with the tribimaximal values [4], given implicitly by $\sin^2\theta_{21} = \frac{1}{3}$, $\sin^2\theta_{32} = \frac{1}{2}$, and $\sin^2\theta_{13} = 0$. The resulting angles are $\theta_{32} = \frac{\pi}{4}\,\mathrm{rad} = 45°$, $\theta_{21} = \sin^{-1}\sqrt{1/3} = 0.6155\ldots\,\mathrm{rad} = 35.2644\ldots°$, and $\theta_{13} = 0$. Explicitly, the tribimaximal mixing matrix is

$$U_{\mathrm{TBM}} = R_{32}\left(\frac{\pi}{4}\right) R_{21}\left(\sin^{-1}\frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 & \sqrt{2} & 0 \\ -1 & \sqrt{2} & \sqrt{3} \\ 1 & -\sqrt{2} & \sqrt{3} \end{pmatrix}. \tag{7}$$

Tribimaximal is a good zeroth order approximation to reality. However, we expect that even if some flavor symmetry is embedded in Nature which leads to zeroth order tribimaximal mixing, in general, there will be deviations from this scheme (see, for example, Ref. [5]).

In this Letter, we present and develop a parametrization of the MNSP matrix which is completely general, but has the tribimaximal matrix as its zeroth order basis. We call the parametrization "triminimal." To accommodate the four independent parameters in U, we introduce as three small quantities ϵ_{jk}, $jk = 21, 32, 13$, the deviations of the

θ_{jk} from their tribimaximal values, and we retain the usual CP-violating phase δ. Tribimaximal mixing, given in Eq. (7), is recovered in the limit of all three $\epsilon_{jk} = 0$. In other parametrizations, all three small parameters are typically coupled in the description of each θ_{jk} [6]. We illustrate the utility of the triminimal parametrization by deriving a rather simple result for the flavor evolution of neutrinos traveling large distances. This includes atmospheric and astrophysical neutrinos. Our new result is presented as an expansion to second order in the three small ϵ_{jk}.

The triminimal parametrization is given by

$$U_{\rm TMin} = R_{32}\left(\frac{\pi}{4}\right)U_{\epsilon}(\epsilon_{32};\epsilon_{13},\delta;\epsilon_{21})R_{21}\left(\sin^{-1}\frac{1}{\sqrt{3}}\right) = \begin{pmatrix}\sqrt{2} & 0 & 0\\ 0 & 1 & 1\\ 0 & -1 & 1\end{pmatrix}\frac{U_\epsilon}{\sqrt{6}}\begin{pmatrix}\sqrt{2} & 1 & 0\\ -1 & \sqrt{2} & 0\\ 0 & 0 & \sqrt{3}\end{pmatrix},$$

$$\text{with } U_\epsilon = R_{32}(\epsilon_{32})U_\delta^\dagger R_{13}(\epsilon_{13})U_\delta R_{21}(\epsilon_{21}) \tag{8}$$

chosen to have just the form of the PDG parametrization. And just as in the PDG parametrization, U_ϵ is unitary, and therefore so is triminimal $U_{\rm TMin}$. The simplicity of Eq. (8) is a fortuitous result of the fact that it is the middle rotation angle (θ_{13}) in the PDG parametrization that is set identically to zero in the tribimaximal scheme.

From Eq. (2)–(5) and (8), one obtains the neutrino mixing observables in terms of the triminimal parameters. The exact result and the expansion up to second order in the ϵ_{jk} are

$$\sin^2\theta_{21} = \frac{1}{3}(\cos\epsilon_{21} + \sqrt{2}\sin\epsilon_{21})^2 \simeq \frac{1}{3} + \frac{2\sqrt{2}}{3}\epsilon_{21} + \frac{1}{3}\epsilon_{21}^2, \tag{9}$$

$$\sin^2\theta_{32} = \frac{1}{2} + \sin\epsilon_{32}\cos\epsilon_{32} \simeq \frac{1}{2} + \epsilon_{32}, \tag{10}$$

$$U_{e3} = \sin\epsilon_{13}e^{-i\delta}. \tag{11}$$

One sees in the above expressions that the triminimal parametrization maintains the simple parametrization of U_{e3}. This is inevitable, for ϵ_{13} and δ are just the standard PDG parameters θ_{13} and δ.

Being a 3-flavor quantity, J_{CP} depends on all three ϵ_{jk}'s. Its expansion is

$$J_{CP} = \frac{\sin\delta}{24}\cos2\epsilon_{32}\sin2\epsilon_{13}\cos\epsilon_{13}(2\sqrt{2}\cos2\epsilon_{21} + \sin2\epsilon_{21}) \simeq \frac{1}{3\sqrt{2}}(\epsilon_{13}\sin\delta)\left(1 + \frac{1}{\sqrt{2}}\epsilon_{21}\right). \tag{12}$$

At lowest order, J_{CP} depends on just $\epsilon_{13}\sin\delta$. J_{CP} has no dependence on ϵ_{32} at first or second order.

In the above formulae, we have truncated the expansions in powers of ϵ_{jk} at quadratic order since cubic order is likely to be immeasurably small. The allowed ranges of the small ϵ_{jk} are obtained from the $(1\sigma)3\sigma$ ranges of the large oscillation angles in Eq. (6). The allowed ranges are

$$-0.08(-0.04) \le \epsilon_{21} \le (0.01)0.07, \tag{13}$$

$$-0.18(-0.10) \le \epsilon_{32} \le (0.04)0.15, \tag{14}$$

$$|\epsilon_{13}| \le (0.14)0.23, \tag{15}$$

while the CP-invariant lies in the range $|J_{CP}| \le (0.03)0.05$.

Let us emphasize two virtues of triminimality. With the ordering of the (small-angle) rotations in Eq. (8) chosen in accord with the PDG parametrization, (i) each ϵ_{jk} is directly interpretable as the deviation of the associated θ_{32}, θ_{13}, or θ_{21} from its tribimaximal value; due to the noncommutivity of rotation matrices, this feature is not shared with other parametrizations of the MNSP matrix [6], but rather is unique to the triminimal parametrization; (ii) the usual PDG result for U_{e3}, namely, $U_{e3} = \sin\epsilon_{13}e^{-i\delta}$, is maintained.

From Eq. (8), it is straightforward to derive the expansion of $U_{\rm TMin}$ in powers of the three ϵ_{jk} and single δ [7,8]. The result is

$$\begin{aligned} U_{\rm TMin} = U_{\rm TBM} &- \frac{\epsilon_{21}}{\sqrt{6}}\begin{pmatrix}\sqrt{2} & -2 & 0\\ \sqrt{2} & 1 & 0\\ -\sqrt{2} & -1 & 0\end{pmatrix} - \frac{\epsilon_{32}}{\sqrt{6}}\begin{pmatrix}0 & 0 & 0\\ -1 & \sqrt{2} & -\sqrt{3}\\ -1 & \sqrt{2} & \sqrt{3}\end{pmatrix} - \frac{\epsilon_{13}}{\sqrt{6}}\begin{pmatrix}0 & 0 & -\sqrt{6}e^{-i\delta}\\ \sqrt{2}e^{i\delta} & e^{i\delta} & 0\\ \sqrt{2}e^{i\delta} & e^{i\delta} & 0\end{pmatrix}\\ &- \frac{\epsilon_{21}^2}{2\sqrt{6}}\begin{pmatrix}2 & \sqrt{2} & 0\\ -1 & \sqrt{2} & 0\\ 1 & -\sqrt{2} & 0\end{pmatrix} - \frac{\epsilon_{32}^2}{2\sqrt{6}}\begin{pmatrix}0 & 0 & 0\\ -1 & \sqrt{2} & \sqrt{3}\\ 1 & -\sqrt{2} & \sqrt{3}\end{pmatrix} - \frac{\epsilon_{13}^2}{2\sqrt{6}}\begin{pmatrix}2 & \sqrt{2} & 0\\ 0 & 0 & \sqrt{3}\\ 0 & 0 & \sqrt{3}\end{pmatrix} - \frac{\epsilon_{21}\epsilon_{32}}{\sqrt{6}}\begin{pmatrix}0 & 0 & 0\\ -\sqrt{2} & -1 & 0\\ -\sqrt{2} & -1 & 0\end{pmatrix}\\ &- \frac{\epsilon_{21}\epsilon_{13}e^{i\delta}}{\sqrt{6}}\begin{pmatrix}0 & 0 & 0\\ -1 & \sqrt{2} & 0\\ -1 & \sqrt{2} & 0\end{pmatrix} - \frac{\epsilon_{32}\epsilon_{13}e^{i\delta}}{\sqrt{6}}\begin{pmatrix}0 & 0 & 0\\ \sqrt{2} & 1 & 0\\ -\sqrt{2} & -1 & 0\end{pmatrix} + \mathcal{O}(\epsilon^3). \end{aligned} \tag{16}$$

In the remainder of this Letter, we present and explore the triminimal parametrization of the phase-averaged mixing that describes atmospheric and astrophysical neutrino flavor propagation. We first introduce the matrix $\underline{U}$ of classical probabilities, defined by $(\underline{U})_{\alpha j} \equiv |U_{\alpha j}|^2$. The full matrix of squared elements, through order $\mathcal{O}(\epsilon^2)$, is

$$\underline{U}_{\mathrm{TMin}} = \frac{1}{6}\left\{\begin{pmatrix} 4 & 2 & 0 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} - E_1\begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix} - E_2\begin{pmatrix} 2 & -2 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} - 2\epsilon_{32}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & -3 \\ -1 & -2 & 3 \end{pmatrix} - \epsilon_{13}^2\begin{pmatrix} 4 & 2 & -6 \\ -2 & -1 & 3 \\ -2 & -1 & 3 \end{pmatrix}\right\}, \tag{17}$$

where $E_1 = 2\sqrt{2}\epsilon_{13}\cos\delta + 2\epsilon_{21}(\epsilon_{13}\cos\delta - 2\sqrt{2}\epsilon_{32})$, and $E_2 = 2\sqrt{2}\epsilon_{21} + \epsilon_{21}^2$. That there are four independent terms in (17) reflects the fact that there are four independent moduli in the neutrino mixing matrix [9].

Some useful results follow immediately from this matrix. For example, if neutrinos are unstable, only the lightest neutrino mass eigenstate arrives at Earth from cosmically-distant sources [10]. Flavor ratios at Earth for the normal mass-hierarchy are $\underline{U}_{e1}$:$\underline{U}_{\mu 1}$:$\underline{U}_{\tau 1}$, and for the inverted mass-hierarchy are $\underline{U}_{e3}$:$\underline{U}_{\mu 3}$:$\underline{U}_{\tau 3}$. These ratios may be read off directly from the 1st and 3rd columns of Eq. (17). As another example, neutrinos emanating from the Sun are nearly pure ν_2 mass states [11]; consequently, their flavor ratios at Earth are mainly given by the 2nd column of (17).

As is well known, the neutrino mixing probabilities for phase-averaged propagation (appropriate when the oscillation phase ($\Delta m^2 L/4E$) is much larger than 1) are given by

$$P_{\nu_\alpha \leftrightarrow \nu_\beta} = \sum_i |U_{\alpha i}|^2 |U_{\beta i}|^2 = (\underline{U}\,\underline{U}^T)_{\alpha\beta}. \tag{18}$$

The full result in terms of the ϵ_{jk} is

$$(\underline{U}\,\underline{U}^T)_{\mathrm{TMin}} = \frac{1}{18}\left\{\begin{pmatrix} 10 & 4 & 4 \\ 4 & 7 & 7 \\ 4 & 7 & 7 \end{pmatrix} + A\begin{pmatrix} 4 & -2 & -2 \\ -2 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} + C\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}\right\} + \mathcal{O}(\epsilon^3), \tag{19}$$

where

$$A(\epsilon_{21}; \epsilon_{21}^2, \epsilon_{13}^2) = -(2\sqrt{2}\epsilon_{21} - 7\epsilon_{21}^2 + 5\epsilon_{13}^2),$$

$$B(\epsilon_{32}, \epsilon_{13}\cos\delta; \epsilon_{21}\epsilon_{32}, \epsilon_{21}\epsilon_{13}\cos\delta) = -2(4\epsilon_{32} - \sqrt{2}\epsilon_{13}\cos\delta + 4\sqrt{2}\epsilon_{21}\epsilon_{32} + 7\epsilon_{21}\epsilon_{13}\cos\delta),$$

$$C(\epsilon_{32}^2, \epsilon_{32}\epsilon_{13}\cos\delta, (\epsilon_{13}\cos\delta)^2) = 4(7\epsilon_{32}^2 + 2(\epsilon_{13}\cos\delta)^2 + \sqrt{2}\epsilon_{32}\epsilon_{13}\cos\delta).$$

The symmetric matrix in Eq. (19) contains the six explicit flavor-mixing probabilities

$$\begin{aligned} P_{\nu_e \leftrightarrow \nu_e} &= \frac{1}{18}(10 + 4A), \\ P_{\nu_\mu \leftrightarrow \nu_\mu} &= \frac{1}{18}(7 + A - B + C), \\ P_{\nu_\tau \leftrightarrow \nu_\tau} &= \frac{1}{18}(7 + A + B + C), \\ P_{\nu_e \leftrightarrow \nu_\mu} &= \frac{1}{18}(4 - 2A + B), \\ P_{\nu_e \leftrightarrow \nu_\tau} &= \frac{1}{18}(4 - 2A - B), \\ P_{\nu_\mu \leftrightarrow \nu_\tau} &= \frac{1}{18}(7 + A - C). \end{aligned} \tag{20}$$

At first sight, it seems remarkable that only three terms, A, B, and C, have emerged to parametrize the six elements $P_{\nu_\alpha \leftrightarrow \nu_\beta}$ in $(\underline{U}\,\underline{U}^T)$. However, this is inevitable, for there are only three independent $P_{\nu_\alpha \leftrightarrow \nu_\beta}$ as a result of the unitary sum rules $\sum_\beta P_{\nu_\alpha \leftrightarrow \nu_\beta} = 1$.

Notice that since each row in $((\underline{U}\,\underline{U}^T))$ partitions a flavor neutrino among all possible flavors, each row must sum to unity at zeroth order in ϵ_{jk}, and to zero at each nonzero order in ϵ_{jk} and $\cos\delta$. Then, because of T-reversal invariance, or equivalently, because $((\underline{U}\,\underline{U}^T))$ is a symmetric matrix, each column must also sum to unity at zeroth order in ϵ_{jk}, and sum to zero at each nonzero order in ϵ_{jk} and $\cos\delta$.

A and B are of indeterminate sign, whereas C, which may be written as $(2\sqrt{2}\epsilon_{13}\cos\delta + \epsilon_{32})^2 + 27\epsilon_{32}^2$, is manifestly positive semidefinite. A and B contain terms linear in ϵ's, as well as quadratic terms; C is purely quadratic in ϵ's. For certain values of the ϵ_{jk}, the second order corrections may dominate the first order corrections. Consequently, C should not be neglected [12].

The dependences of flavor oscillation and flavor-mixing probabilities on first and second order corrections will be examined in considerable detail, for low and high energy neutrinos, in [13]. Here, we present an interesting application of the phase-averaged mixing matrix given in Eq. (19). The most common source for atmospheric and astrophysical neutrinos is thought to be pion production and decay. The pion decay chain generates an initial neutrino flux with flavor composition given approximately [14] by Φ_e^0:Φ_μ^0:Φ_τ^0 = 1:2:0 for the neutrino fluxes. According to

Eq. (19), the fluxes Φ_α arriving at Earth have a flavor ratio of

$$\Phi_e{:}\Phi_\mu{:}\Phi_\tau = 1+\frac{1}{18}(2B){:}1-\frac{1}{18}(B-2C){:}1-\frac{1}{18}(B+2C). \tag{21}$$

Violation of $\nu_\mu \leftrightarrow \nu_\tau$ symmetry, exact with tribimaximal mixing, is directly assessed via the flavor ratio

$$\frac{\Phi_\mu}{\Phi_\tau} = 1 + \frac{2}{9}C + \mathcal{O}(\epsilon^3). \tag{22}$$

From this result, we may infer two lessons: $C \geq 0$, so we learn that $\Phi_\mu \geq \Phi_\tau$ is an inevitable consequence for pion-produced, astrophysical neutrinos; and we see an explicit example where the second order correction dominates over the first order correction (exactly zero in this case). The triminimal parametrization and expansion has made this result transparent.

In summary, we have presented the triminimal parametrization of the MNSP matrix. Three small parameters $\epsilon_{jk} = \theta_{jk} - (\theta_{jk})_{\rm TBM} \ll 1$, each equal to the deviation of one of the measured quantities θ_{jk} from its tribimaximal value, plus the usual CP-violating phase δ, comprise the parametrization. The triminimal parametrization leads to simple formulas for neutrino flavor mixing. The proposed parametrization in Eq. (8), the expansions in (16), (17), and (19), and the mixing probabilities in (20) are the main results of this Letter. Simple properties of the triminimal parametrization are not shared by other parametrizations.

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Note added.—As this Letter was being written, a very similar proposal for parametrization was posted on the ArXiv [15]. There, the utility of triminimal parameters for terrestrial flavor oscillations was emphasized; here, we emphasize the utility for phase-averaged atmospheric and astrophysical flavor mixing. After this Letter was written, we were made aware of some earlier but different parametrizations of ‘‘almost tribimaximal’’ mixing matrices [16].

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[1] R. N. Mohapatra *et al.*, Rep. Prog. Phys. **70**, 1757 (2007); R. N. Mohapatra and A. Y. Smirnov, Annu. Rev. Nucl. Part. Sci. **56**, 569 (2006); A. Strumia and F. Vissani, arXiv:hep-ph/0606054.

[2] W. M. Yao *et al.* (Particle Data Group), J. Phys. G **33**, 1 (2006).

[3] M. C. Gonzalez-Garcia and M. Maltoni, arXiv:0704.1800.

[4] P. F. Harrison, D. H. Perkins, and W. G. Scott, Phys. Lett. B **530**, 167 (2002); **535**, 163 (2002); Z. Z. Xing, *ibid.* **533**, 85 (2002); X. G. He and A. Zee, *ibid.* **560**, 87 (2003); see also L. Wolfenstein, Phys. Rev. D **18**, 958 (1978); Y. Yamanaka, H. Sugawara, and S. Pakvasa, Phys. Rev. D **25**, 1895 (1982); **29**, 2135(E) (1984).

[5] F. Plentinger and W. Rodejohann, Phys. Lett. B **625**, 264 (2005); S. Antusch and S. F. King, Phys. Lett. B **631**, 42 (2005); S. Luo and Z. Z. Xing, Phys. Lett. B **632**, 341 (2006); N. Haba, A. Watanabe, and K. Yoshioka, Phys. Rev. Lett. **97**, 041601 (2006); M. Hirsch, E. Ma, J. C. Romao, J. W. F. Valle, and A. Villanova del Moral, Phys. Rev. D **75**, 053006 (2007); X. G. He and A. Zee, Phys. Lett. B **645**, 427 (2007); A. Dighe, S. Goswami, and W. Rodejohann, Phys. Rev. D **75**, 073023 (2007); S. Antusch, P. Huber, S. F. King, and T. Schwetz, J. High Energy Phys. 04 (2007) 060; M. Lindner and W. Rodejohann, J. High Energy Phys. 05 (2007) 089; K. A. Hochmuth, S. T. Petcov, and W. Rodejohann, Phys. Lett. B **654**, 177 (2007).

[6] A. Zee, Phys. Rev. D **68**, 093002 (2003); W. Rodejohann, Phys. Rev. D **69**, 033005 (2004); N. Li and B. Q. Ma, Phys. Rev. D **71**, 017302 (2005); A. Datta, L. Everett, and P. Ramond, Phys. Lett. B **620**, 42 (2005); L. L. Everett, Phys. Rev. D **73**, 013011 (2006); J. D. Bjorken, P. F. Harrison, and W. G. Scott, Phys. Rev. D **74**, 073012 (2006).

[7] First order expansions in ϵ_{21} and ϵ_{32} may be found in R. Lehnert and T. J. Weiler, arXiv:0708.1035.

[8] In deriving this formula, it is useful to expand in powers of ϵ the following form of the individual rotation matrices:

$$R(\epsilon) = \begin{pmatrix} \cos\epsilon & \sin\epsilon \\ -\sin\epsilon & \cos\epsilon \end{pmatrix} = \mathbb{1} - 2\sin\frac{\epsilon}{2}\begin{pmatrix} \sin\frac{\epsilon}{2} & -\cos\frac{\epsilon}{2} \\ \cos\frac{\epsilon}{2} & \sin\frac{\epsilon}{2} \end{pmatrix}.$$

[9] In fact, any four of the parametrization-invariant moduli have been proposed as the fundamental building blocks of the leptonic-mixing edifice; see P. F. Harrison, W. G. Scott, and T. J. Weiler, Phys. Lett. B **641**, 372 (2006).

[10] S. Pakvasa, Lett. Nuovo Cimento Soc. Ital. Fis. **31**, 497 (1981); J. F. Beacom, N. F. Bell, D. Hooper, S. Pakvasa, and T. J. Weiler, Phys. Rev. Lett. **90**, 181301 (2003); Phys. Rev. D **69**, 017303 (2004), and references therein.

[11] S. J. Parke, H. Nunokawa, and R. Zukanovich Funchal, AIP Conf. Proc. **842**, 889 (2006).

[12] A ‘‘universal’’ parameter related to B has been noted previously in the literature. It is called $-\Delta$ in Z. Z. Xing, Phys. Rev. D **74**, 013009 (2006); $+\Delta$ in W. Rodejohann, J. Cosmol. Astropart. Phys. 01 (2007) 029.

[13] S. Pakvasa, W. Rodejohann, and T. J. Weiler, J. High Energy Phys. 02 (2008) 005.

[14] P. Lipari, M. Lusignoli, and D. Meloni, Phys. Rev. D **75**, 123005 (2007).

[15] S. F. King, Phys. Lett. B **659**, 244 (2008).

[16] Z. Z. Xing, Phys. Lett. B **533**, 85 (2002); S. Parke, in *“20th Int’l. Weak Interaction and Neutrinos,” Delphi, Greece, June 6–11, 2005*, http://conferences.phys.uoa.gr/win05/; P. F. Harrison, in *“3rd Int’l. Scoping Study of a Future Neutrino Factory and Super-beam Facility,” April 24–28, 2006* (Rutherford-Appleton Lab, United Kingdom, 2006).