

Nonmaximally Entangled States Can Be Better for Multiple Linear Optical Teleportation

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We investigate multiple linear optical teleportation in the Knill-Laflamme-Milburn scheme with both maximally and nonmaximally entangled states. We show that if the qubit is teleported several times via a nonmaximally entangled state, then the errors introduced in the previous teleportations can be corrected by the errors introduced in the following teleportations. This effect is so strong that it leads to another interesting phenomenon: i.e., the total probability of successful multiple linear optical teleportation is higher for nonmaximally entangled states than maximally entangled states.

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One of the main activities in quantum computation field is linear optical processing of quantum information [1]. In particular, the very first experimental demonstration of quantum teleportation was based on linear optics [2]. However, the teleportation only had a success probability of 25% partially due to the impossibility of performing complete Bell measurement [3]. In order to perform scalable quantum computation, it is of essential importance to improve this success probability to a value close to 100%. Recently, Knill, Laflamme, and Milburn (KLM) [4] have shown that the probability of success for the teleportation of a superposition of vacuum and one photon Fock state can indeed be increased by using a maximally entangled state of two $N + 1$ dimensional Hilbert spaces encoded in N photons. The probability that teleportation succeeds is then equal to $1 - \frac{1}{N+1}$. Moreover, when teleportation succeeds, the fidelity of the teleported qubit is equal to 1. Spedalieri *et al.* [5] generalized their protocol for polarization encoding of a qubit, i.e., when one uses horizontal and vertical polarizations rather than photon number states to represent the logical values 0 and 1. Franson *et al.* [6] proposed a different scheme, which does not require that a qubit has to be teleported with the perfect fidelity but rather assumes that the qubit is always teleported successfully and aims at maximizing the average fidelity. Their scheme is based in fact on the KLM scheme with another carefully chosen entangled state. Franson *et al.* have shown that their scheme gives better average fidelity of the teleported qubit than the KLM scheme. In [7], we have shown that if the aim is maximization of the probability of successful teleportation and one requires unit fidelity of the teleported qubit, then the state used in the original KLM scheme is optimal. Thus, the maximally entangled state is best suited for single quantum teleportation.

In this Letter, we consider several subsequent linear optical teleportations; i.e., the qubit is teleported from A to C , then from C to D , and so on, and finally to B . We show an interesting phenomenon that when the final unit

fidelity of the teleported qubit is required after completion of all teleportations, then the nonmaximally entangled states give higher probability of successful teleportation than the maximally entangled ones. It is surprising because usually maximally entangled states are optimal for information-theoretical tasks [8].

Let us begin with description of a generalization of the KLM scheme of linear optical teleportation to the one which is based on the nonmaximally entangled states [6]. In this scheme, one uses the following entangled state

$$|t_N\rangle = \sum_{i=0}^N c_i |V\rangle^i |H\rangle^{N-i} |H\rangle^i |V\rangle^{N-i}, \quad (1)$$

where $|V\rangle^i$ stands for $|V\rangle_1 |V\rangle_2 \dots |V\rangle_i$, i.e., one vertically polarized photon in each of the subsequent modes. Similarly, $|H\rangle^{N-i}$ stands for $|H\rangle_{i+1} |H\rangle_{i+2} \dots |H\rangle_N$, i.e., one horizontally polarized photon in each of the subsequent modes. If we use the states $\{|V\rangle^i |H\rangle^{N-i}; i = 0, 1, \dots, N\}$ and $\{|H\rangle^i |V\rangle^{N-i}; i = 0, 1, \dots, N\}$ as the orthonormal basis states of Alice's and Bob's Hilbert spaces, respectively, we may treat the state of Eq. (1) as a bipartite entangled state. For $c_i = \frac{1}{\sqrt{N+1}}$, we obtain a maximally entangled state. In order to teleport a qubit in the state $|\psi\rangle = \alpha|H\rangle + \beta|V\rangle$, Alice applies the $(N + 1)$ -point quantum Fourier transform to the input mode and the N first modes of the state $|t_N\rangle$, which is given by

$$F_N(v_k^\dagger) = \frac{1}{\sqrt{N+1}} \sum_{l_k=0}^N \omega^{kl_k} v_{l_k}^\dagger, \quad (2)$$

$$F_N(h_k^\dagger) = \frac{1}{\sqrt{N+1}} \sum_{l_k=0}^N \omega^{kl_k} h_{l_k}^\dagger.$$

In the above equations, v_k^\dagger and h_k^\dagger are the creation operators for vertically and horizontally polarized photons in mode k , respectively, and $\omega = e^{i2\pi/(N+1)}$. After this transformation, the state of the system is

$$\left(\frac{1}{\sqrt{N+1}}\right)^{N+2} \sum_{i=0}^N \sum_{l_0, \dots, l_N=0}^N \left(\omega^{\sum_{k=0}^N k l_k} \alpha h_{l_0}^\dagger v_{l_1}^\dagger \dots v_{l_i}^\dagger h_{l_{i+1}}^\dagger \dots h_{l_N}^\dagger + \beta v_{l_0}^\dagger v_{l_1}^\dagger \dots v_{l_i}^\dagger h_{l_{i+1}}^\dagger \dots h_{l_N}^\dagger \right) c_i |VAC\rangle_{0\dots N} |H\rangle^i |V\rangle^{N-i}. \quad (3)$$

Note that in the first term, there are i creation operators for vertically polarized photons while in the second term, there are $i+1$ creation operators for vertically polarized photons. Next, Alice measures the total number of vertically polarized photons and horizontally polarized photons in each of the first $N+1$ modes. If she detects v_j vertically polarized photons and h_j horizontally polarized photons in mode j , then the state of the last N modes is

$$|H\rangle^{m-1} \frac{1}{\sqrt{p(m)}} \left(\alpha c_m |H\rangle + \beta c_{m-1} \omega^{-\sum_{j=0}^N j(v_j+h_j)} |V\rangle \right) |V\rangle^{N-m}, \quad (4)$$

where $0 < m = \sum_{j=0}^N v_j < N+1$ is the total number of vertically polarized photons detected and $p(m) = |\alpha c_m|^2 + |\beta c_{m-1}|^2$ is the total probability of detecting m vertically polarized photons. However, if Alice detects 0 or $N+1$ vertically polarized photons, then the state of the qubit is irreversibly destroyed, which happens with the average probability of $\frac{1}{2}(|c_0|^2 + |c_N|^2)$. One can see that the modified state of the teleported qubit is found in the $N+m$ th mode. After correction of the phase, this state becomes

$$|\psi_m\rangle = \frac{1}{\sqrt{p(m)}} (\alpha c_m |H\rangle + \beta c_{m-1} |V\rangle). \quad (5)$$

The qubit can be returned to its original state by performing the generalized measurement given by the Kraus operators:

$$E_S = \frac{c_{m-1}}{c_m} |H\rangle\langle H| + |V\rangle\langle V|, \quad (6)$$

$$E_F = \sqrt{1 - \left|\frac{c_{m-1}}{c_m}\right|^2} |H\rangle\langle H|,$$

for $|c_{m-1}| \leq |c_m|$. A similar measurement exists if $|c_{m-1}| > |c_m|$. When E_S is applied, then the qubit ends in its original state $|\Psi\rangle = \alpha|H\rangle + \beta|V\rangle$. The probability of successful error correction is

$$p(S|m) = \langle \psi_m | E_S^\dagger E_S | \psi_m \rangle = \frac{|c_{m-1}|^2}{p(m)}. \quad (7)$$

In [7], we described how such a measurement can be implemented experimentally with linear optics. In order to obtain the total probability of a successful teleportation, we have to sum up the joint probabilities of detecting m vertically polarized photons and a successful error correction. Let us recall that if 0 or $N+1$ photons are detected, then the teleportation fails. Hence, we restrict the summation over m from 1 to N . We obtain

$$p(S) = \sum_{m=1}^N p(S, m) = \sum_{m=1}^N p(S|m) p(m) = \sum_{m=1}^N \min\{|c_{m-1}|^2, |c_m|^2\}. \quad (8)$$

Let us now suppose that the qubit is to be teleported once again (see Fig. 1). Then the simplest strategy is to perform the first teleportation followed by the error correction, then the second teleportation followed by the error correction. However, it is not the optimal strategy. Let us, thus, assume that we do not correct the error introduced in the first teleportation and teleport the qubit once again with the use of the identical entangled state. After the second teleportation, the state of the qubit is

$$|\psi_{m,n}\rangle = \frac{1}{\sqrt{p(m,n)}} (\alpha c_m c_n |H\rangle + \beta c_{m-1} c_{n-1} |V\rangle), \quad (9)$$

where $p(m,n)$ is the joint probability of detecting m vertically polarized photons in the first teleportation and n vertically polarized photons in the second teleportation, and is given by

$$p(m,n) = p(n|m) p(m) = |\alpha c_m c_n|^2 + |\beta c_{m-1} c_{n-1}|^2. \quad (10)$$

If $c_m = c_{n-1}$ and $c_n = c_{m-1}$, then the state of the qubit is

$$|\psi_{m,n}\rangle = \alpha |H\rangle + \beta |V\rangle; \quad (11)$$

i.e., it is the original state of the qubit, and we do not have to perform the error correction. The second teleportation corrected the error introduced by the first teleportation. We call this effect the error self-correction. A similar effect occurs for entanglement swapping as considered by Acin, Cirac, and Lewenstein [9] (see also: [10–13]). In general, if $|c_{m-1} c_{n-1}| < |c_m c_n|$, then one can recover the original state of the qubit by performing generalized measurement given by the Kraus operators:

$$E_S = \frac{c_{m-1} c_{n-1}}{c_m c_n} |H\rangle\langle H| + |V\rangle\langle V|, \quad (12)$$

$$E_F = \sqrt{1 - \left|\frac{c_{m-1} c_{n-1}}{c_m c_n}\right|^2} |H\rangle\langle H|.$$

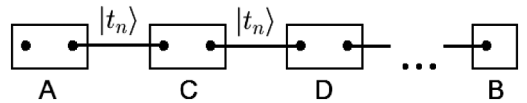


FIG. 1. Several subsequent teleportations. A better strategy is not to perform error correction at C , D , and so on but to perform it at B after completion of all teleportations.

A similar measurement exists if $|c_{m-1}c_{n-1}| > |c_m c_n|$. The joint probability of detecting m vertically polarized photons in the first teleportation and n vertically polarized photons in the second teleportation and the successful error correction is

$$p(S, m, n) = \min(|c_m c_n|^2, |c_{m-1} c_{n-1}|^2). \quad (13)$$

On the other hand, if we performed the first teleportation followed by the error correction and the second teleportation followed by the error correction, then the probability of detecting m vertically polarized photons in the first teleportation and n vertically polarized photons in the second teleportation and the successful correction of both errors would be

$$p(S, m_1, m_2, \dots, m_M) = \min(|c_{m_1} c_{m_2} \dots c_{m_M}|^2, |c_{m_1-1} c_{m_2-1} \dots c_{m_M-1}|^2). \quad (15)$$

In order to obtain the total probability of successful multiple teleportation, we have to sum these probabilities over m_1, m_2, \dots, m_M ranging from 1 to N . We obtain

$$p(S) = \sum_{m_1=1, m_2=1, \dots, m_M=1}^N \min(|c_{m_1} c_{m_2} \dots c_{m_M}|^2, |c_{m_1-1} c_{m_2-1} \dots c_{m_M-1}|^2). \quad (16)$$

Let us now take the following six-photon entangled state whose coefficients c_i depend on the parameter x (see Fig. 2)

$$|t_6\rangle = \sum_{i=0}^6 \sqrt{\frac{1-9x}{7} + (3-|i-3|x)|V\rangle^i |H\rangle^{6-i} |H\rangle^i |V\rangle^{6-i}}. \quad (17)$$

The coefficients are symmetric around $i=3$, and the parameter x is the slope of the line connecting the points $(i, |c_i|^2)$. For $x=0$, the state is maximally entangled. Note that for $x>0$, the smallest coefficients are c_0 and c_6 . On the other hand, the average probability that the state of the teleported qubit will be irreversibly destroyed during teleportation (and before error correction) is $\frac{1}{2}(|c_0|^2 + |c_6|^2)$. We can lower this probability by lowering the coefficients c_0 and c_6 . We should remember that in such a case, we

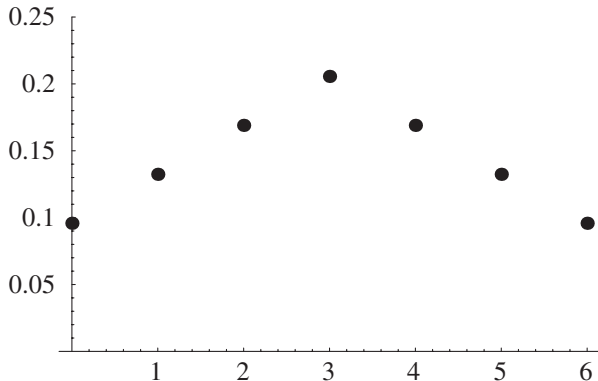


FIG. 2. Exemplary series of coefficients $|c_i|^2$ of entangled state of Eq. (17) ($x = 0.0366$).

$$p'(S, m, n) = \min(|c_m|^2, |c_{m-1}|^2) \min(|c_n|^2, |c_{n-1}|^2), \quad (14)$$

which is lower or equal to the previous probability. Moreover, if $(|c_{m-1}| > |c_m|$ and $|c_{n-1}| < |c_n|)$ or $(|c_{m-1}| < |c_m|$ and $|c_{n-1}| > |c_n|)$, then the probability $p'(S, m, n)$ is lower than the probability $p(S, m, n)$. We conclude that it is better to perform the error correction at the end when all teleportations were completed.

Let us now suppose that we perform M subsequent teleportations with the use of the identical entangled states of Eq. (1). A straightforward calculation gives the following probability of detecting m_1, m_2, \dots, m_M vertically polarized photons in the first, second, \dots , M th teleportation, and the final successful error correction

increase the probability that the state will be irreversibly destroyed during error correction. Let us calculate with the help of Eq. (16) the total probability of successful six subsequent teleportations of a qubit with the error correction at the end.

In Fig. 3, we present how the probability of successful multiple teleportation depends on the parameter x . For $x=0$, the probability of successful teleportation is $p=0.3965$. However, for $x \geq 0$, this probability slowly increases with x reaching its maximal value $p=0.4152$ for $x=0.0366$. Hence, we obtain an interesting phenomenon—the probability of the successful multiple teleportation is greater for nonmaximally entangled states than for maximally entangled ones. It should be compared with the probability of successful single teleportation which reaches always its maximal value for maximally entangled state [7]. Let us also point out that the probability for $x=0$

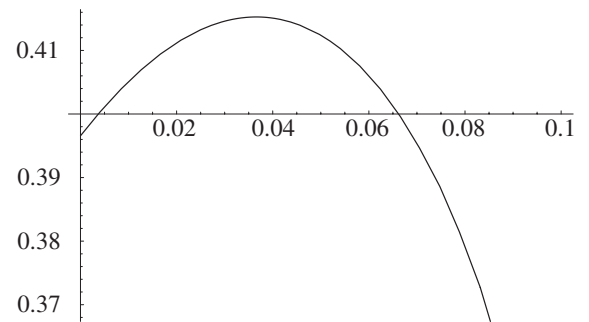


FIG. 3. Probability of successful six teleportations with entangled state of Eq. (17) as a function of x . For $x=0$, the state is maximally entangled while for $x \neq 0$, it is nonmaximally entangled.

is equal to the product of probabilities of each successful teleportation, i.e., $p = (\frac{6}{9})^6 = 0.3965$. We can see that if we use nonmaximally entangled states, we have an increase in the probability $p(0.0366) - p(0) = 0.0187$. The relative increase in the probability is $\frac{p(0.0366) - p(0)}{p(0)} = 0.0471$. Thus, the use of nonmaximally entangled states contributes in about 5% to the total probability. The effect may be even stronger when one increases the number of photons in the entangled state and/or the number of subsequent teleportations. It is also interesting to calculate the probability $p'(0.0366)$ of successful teleportation when one performs error correction between subsequent teleportations. We obtain $p'(0.0366) = 0.2511$. This probability is lower than the probability of successful teleportation with the final error correction $p(0.0366) = 0.4152$. The increase in the probability due to the error self-correction is $p(0.0366) - p'(0.0366) = 0.1641$, and the relative increase is $\frac{p(0.0366) - p'(0.0366)}{p'(0.0366)} = 0.6535$.

Let us now have a look at the origin of this effect. The teleportation does not succeed when one of the senders detects 0 or $N + 1$ vertically polarized photons which happens with the average probabilities of $\frac{1}{2}|c_0|^2$ and $\frac{1}{2}|c_N|^2$, respectively. We can decrease these probabilities by decreasing the coefficients c_0 and c_N . If we do it and perform single teleportation, then the error correction is needed. The probabilistic nature of the error correction decreases the total probability of successful teleportation, and there is no gain. However, if we perform several teleportations with no error correction between subsequent teleportations, then the error self-correction may occur. This error self-correction may correct the errors or increase the probability of a successful error correction at the end. This effect allows us to obtain higher probability of successful teleportations with nonmaximally entangled states which have smoothly lowered the probabilities of having 0 and N vertically polarized photons.

In summary, we have considered several subsequent teleportations of a qubit in the KLM scheme. We have shown how the errors introduced in the previous teleportations can be corrected by the errors introduced in the following teleportations. This effect leads to an interesting new phenomenon. Namely, nonmaximally entangled states can be better for multiple linear optical teleportation. This strange behavior is connected to the fact that with linear optics, one cannot perform the complete Bell measure-

ment, and hence, quantum teleportation can be implemented only probabilistically [3]. We believe that our research will lead to deeper understanding of manipulation of entanglement with local linear optical operations and classical communication which are a natural analog of local operations and classical communication usually considered in entanglement theory. Our result may have applications in linear optical quantum computation and quantum networks.

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