



Completeness of the Classical 2D Ising Model and Universal Quantum Computation

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We prove that the 2D Ising model is complete in the sense that the partition function of any classical q -state spin model (on an arbitrary graph) can be expressed as a special instance of the partition function of a 2D Ising model with complex inhomogeneous couplings and external fields. In the case where the original model is an Ising or Potts-type model, we find that the corresponding 2D square lattice requires only polynomially more spins with respect to the original one, and we give a constructive method to map such models to the 2D Ising model. For more general models the overhead in system size may be exponential. The results are established by connecting classical spin models with measurement-based quantum computation and invoking the universality of the 2D cluster states.

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I. Introduction.—Classical spin models such as the Ising and Potts models are widely studied in statistical physics, because they provide important toy models for magnetism and they can be mapped to numerous interesting problems in physics and mathematics [1,2]. The geometry of a model, in particular its spatial dimension, plays an important role with respect to the physical properties of the system and the possibility of finding (approximate) solutions. For instance, it is known that evaluation of the partition function of the Ising model with magnetic fields is easy in 1D, while on a 2D square lattice this problem is already NP-hard [3].

In this Letter we study the interrelations between classical q -state spin models on different geometries (or graphs) and find that the 2D Ising model (which has $q = 2$) plays a distinguished role in this study. We consider mappings that leave the partition function—and hence all thermodynamical quantities, such as free energy or magnetization, derived from it—invariant (see also [4,5]). As the main result of this Letter, we prove that the 2D Ising model is complete in the sense that the partition function of any classical q -state spin model can be expressed as a special instance of the partition function of a 2D Ising model with inhomogeneous couplings. More precisely, given a partition function Z_G of a q -state spin model on an arbitrary graph—which may be, e.g., a lattice of arbitrary dimension or involve long-range interactions—there exists a 2D square lattice of enlarged size, and suitably tuned nearest-neighbor coupling strengths and magnetic fields, such that the partition function of the Ising model on this lattice specializes to Z_G . Furthermore, in the case where the original model on the graph G is an Ising or Potts-type model, we find that the corresponding 2D square lattice requires only polynomially more spins with respect to the original one. For more general models the overhead in system size may be exponential. However, one important remark needs to be made: in order to achieve this result, one has to allow for complex couplings in the 2D

partition function—thus leaving the “physical” regime of the model.

The results are proven by relating the problem at hand to insights from quantum information theory, more particularly to the area of measurement-based quantum computation (MQC). The latter is a recently established paradigm for quantum computation where quantum information is processed by performing sequences of single-qubit measurements on a highly entangled resource state [6]. In order to obtain our results, we first prove that the Ising partition function on an arbitrary graph (with external field) can be written as the overlap between an entangled quantum state and a complete product state—thus generalizing a construction which we introduced in Ref. [5]; see also Ref. [7]. This formulation allows us to make a connection with MQC. In particular, we prove that the entangled state corresponding to the Ising model on a 2D square lattice is (a variant of) the 2D cluster state [8]. The latter is known to be a universal resource state for MQC in the sense that every quantum state can be obtained by performing a suitable sequence of single-qubit measurements on a sufficiently large 2D cluster state. This quantum universality feature of the 2D cluster states leads to the result that the 2D Ising model is complete in the sense specified above.

II. Classical Ising model.—We consider the classical Ising model involving N two-state spins $(s_1, s_2, \dots, s_N) \equiv \mathbf{s}$, where $s_a = \pm 1$. The spins interact pairwise according to an interaction pattern specified by a graph $G = (V, E)$ with vertex set V and edge set E , and the coupling strengths are denoted by J_{ab} . Moreover, the spins are subjected to local magnetic field terms h_a . The Hamiltonian of the system is given by $H_G(\mathbf{s}) := -\sum_{\{a,b\} \in E} J_{ab} s_a s_b - \sum_{a \in V} h_a s_a$. In other words, we consider a general inhomogeneous Ising model on an arbitrary graph. The partition function Z_G is defined by $Z_G(\{J_{ab}, h_a\}) := \sum e^{-\beta H_G(\mathbf{s})}$, where $\beta = (k_B T)^{-1}$, with k_B the Boltzmann constant and T the temperature.

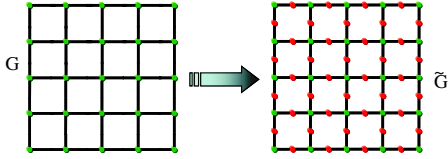


FIG. 1 (color online). Decorated graph \tilde{G} (right) corresponding to a 2D lattice G (left). Dark (green) dots indicate vertices originating from the vertices V of G , while light (red) dots indicate vertices V_E originating from edges E of G .

III. Quantum formulation.—We now show how the partition function Z_G can be expressed in a quantum physics language. Let \tilde{G} be the graph with $n = |V| + |E|$ vertices and $2|E|$ edges which is obtained from G by adding at each edge $\{a, b\} \in E$ an additional vertex ab and thus “splitting every edge in half” (see Fig. 1). We will call \tilde{G} the decorated version of G . The vertex set of \tilde{G} is thus given by the union of the original vertex set V of G and the set $V_E = \{ab\}_{\{a, b\} \in E}$ corresponding to edges of G —note that we label the vertices in V_E by double indices, indicating their origin in the corresponding edge of G . We now consider an n -qubit state $|\varphi_{\tilde{G}}\rangle$, defined on a set of qubits labeled by $V \cup V_E$, which is defined to be the graph state [9,10] associated with the decorated graph \tilde{G} . In particular, $|\varphi_{\tilde{G}}\rangle$ is the (unique) joint fixed point of the $|V| + |E|$ stabilizing operators K_a and K_{ab} ,

$$K_a = X^{(a)} \prod_{b:\{a,b\} \in E} X^{(ab)} \quad K_{ab} = Z^{(ab)} Z^{(a)} Z^{(b)}, \quad (1)$$

for every $a \in V$, and for every $e = \{a, b\} \in E$. Here X and Z denote the Pauli spin matrices, and the notation $X^{(i)}$ ($Z^{(i)}$) indicates that the operator X (Z) acts on qubit i .

We can now express the partition function as follows:

$$Z_G(\{J_{ab}, h_a\}) = 2^{|V|/2} \langle \alpha | \varphi_{\tilde{G}} \rangle. \quad (2)$$

In this expression,

$$|\alpha\rangle = \left(\bigotimes_{ab \in V_E} |\alpha_{ab}\rangle \bigotimes_{a \in V} |\alpha_a\rangle \right) \quad (3)$$

is a complete product state specifying the coupling strengths of the Ising model. In particular, $|\alpha_{ab}\rangle = e^{\beta J_{ab}} |0\rangle + e^{-\beta J_{ab}} |1\rangle$ is an (unnormalized) one-qubit state (acting on qubit ab) determined by the interaction strength between particles a and b . Similarly, $|\alpha_a\rangle = e^{\beta h_a} |0\rangle + e^{-\beta h_a} |1\rangle$ is an (unnormalized) one-qubit state (acting on qubit a) determined by the local magnetic field at particle a . Expression (2) shows that Z_G can be obtained by calculating the inner product of the graph state $|\varphi_{\tilde{G}}\rangle$ and a complete product state. The choice of the product state allows one to specify the couplings of the Hamiltonian and the temperature, while the structure of the graph state reflects the interaction pattern.

To show that Eq. (2) holds, we use that $|\varphi_{\tilde{G}}\rangle$ can be written as $|\varphi_{\tilde{G}}\rangle \propto \sum_t |B^T t\rangle |t\rangle$, where t is a binary vector of

length $|V|$, B is the incidence matrix of the graph G , and by writing out the sum $\langle \alpha | \varphi_{\tilde{G}} \rangle$. The construction of $|\varphi_{\tilde{G}}\rangle$ can be viewed as a generalization of the one we introduced in Ref. [5]. While in Ref. [5] each qubit was associated with an edge of the graph G , here we have two types of vertices: one subset V_E associated to edges (“edge qubits”) and one to vertices V (“vertex qubits”). This enlarging of the system size allows one to also treat local terms in the Hamiltonian (whereas Ref. [5] only dealt with zero external field). In addition, the stabilizer of the state $|\varphi_{\tilde{G}}\rangle$ can be immediately obtained from the graph G describing the interaction pattern (or its decorated version \tilde{G}), as in Eq. (1) and Fig. 1.

IV. MQC and the 2D cluster states.—We now turn our attention to measurement-based (or “one-way”) quantum computation, and establish a relation to the partition function of the 2D classical Ising model via Eq. (2).

The one-way quantum computer [6] is a recently developed model for quantum computation, where computations are realized by performing single-qubit measurements on a highly entangled substrate state called the 2D cluster state $|\mathcal{C}\rangle$ [8]; the latter is a graph state [10] associated to a 2D square lattice \mathcal{C} .

A particular feature of the one-way quantum computer is that it is universal. This means that any n -qubit quantum state can be prepared, up to local unitary Pauli operations, by performing sequences of single-qubit measurements on a $d \times d$ cluster state $|\mathcal{C}\rangle$ of sufficiently large system size $M = d^2$. This property of the 2D cluster states immediately implies that every n -qubit quantum state $|\psi\rangle$ can be written in the following way:

$$\Sigma |\psi\rangle = 2^{(M-n)/2} (I \otimes \langle \beta |) |\mathcal{C}\rangle. \quad (4)$$

This formula represents one “measurement branch” of a one-way computation performed on an M -qubit cluster state, yielding the state $|\psi\rangle$ (up to a local operation Σ) as an output state on the subset of qubits which has not been measured. The dual product state $\langle \beta | = \otimes_j \langle \beta_j |$, which acts only on the measured qubits, is determined by the bases and the outcomes of the different steps in the computation. The local unitary operator Σ (“correction operator”) acts on the unmeasured qubits (i.e., on the Hilbert space of $|\psi\rangle$); the tensor factors of Σ are always instances of Pauli operators: $\Sigma_i \in \{I, X, Y, Z\}$. The prefactor $2^{(M-n)/2}$ reflects the fact that the success probability of every measurement branch is 2^{n-M} .

As proved in Ref. [6], for all n -qubit states $|\psi\rangle$ that can be efficiently prepared in the circuit model, i.e., by a polynomial sequence of two-qubit gates, the required size M of the cluster state in Eq. (4) scales polynomially with the number of qubits: $M \propto \text{poly}(n)$. Moreover, in this case the measurement bases $|\beta_j\rangle$ as well as the correction operations Σ can be efficiently determined. Since any graph state on n qubits can be prepared using at most $O(n^2)$ controlled-phase gates [10], it follows that an arbitrary n -qubit graph state [9] can be written in the form (4)

with $M = \text{poly}(n)$. Furthermore, for the preparation of graph states every single-qubit state $|\beta_j\rangle$ can always be chosen to be one of the X , Y , and Z eigenstates.

Also $|\varphi_{\tilde{C}}\rangle$ (i.e., the state $|\varphi_{\tilde{C}}\rangle$ where $G \equiv C$ is the 2D square lattice) is a universal resource. This is because the 2D-cluster state $|C\rangle$ can be deterministically generated from $|\varphi_{\tilde{C}}\rangle$ (up to a local correction) by performing single-qubit Y measurements on all qubits in V_E . This fact was already noted in [7]. As a consequence, one has

$$\Sigma'|C\rangle = 2^{|E|/2}(I \otimes \langle 0_Y|^{V_E})|\varphi_{\tilde{C}}\rangle, \quad (5)$$

where $|0_Y\rangle^{V_E}$ is a tensor product of the $(+1)$ eigenstate of Y on all edge qubits, and Σ' is a local correction.

V. Universality of the 2D Ising model.—We are now ready to establish the connection between the evaluation of Ising partition functions and universal MQC. With this in mind, consider the Ising model on a graph G . The partition function Z_G can be expressed in the form (2). Now consider the following procedure.

First, the graph state $|\varphi_{\tilde{C}}\rangle$ is written in the form (4) when taking $|\psi\rangle \equiv |\varphi_{\tilde{C}}\rangle$. Together with Eq. (5), this implies that the partition function Z_G can be written as

$$Z_G(\{J_{ab}, h_a\}) = A \cdot \langle \gamma | \varphi_{\tilde{C}} \rangle, \quad (6)$$

where A is a constant and $|\gamma\rangle$ is a product state, $|\gamma\rangle = \Sigma|\alpha\rangle \otimes \Sigma'|\beta\rangle \otimes |0_Y\rangle^{V_E}$. Note that, as $|\varphi_{\tilde{C}}\rangle$ is a graph state, the system size of the 2D cluster state grows polynomially with the size of G . Furthermore, $|\beta\rangle$ consists of X , Y , and Z eigenstates.

Now, applying Eq. (2) to the 2D Ising model, the overlap between $|\varphi_{\tilde{C}}\rangle$ and a complete product state corresponds to a 2D Ising partition function Z_{2D} , evaluated in certain couplings $\{J'_{ij}, h'_i\}$ determined by $|\gamma\rangle$. This allows us to conclude that Z_G can be written as follows:

$$Z_G(\{J_{ab}, h_a\}) \propto Z_{2D}(\{J'_{ij}, h'_i\}). \quad (7)$$

In other words, the Ising partition function on an arbitrary graph can be recovered as a special instance of the Ising partition function on a 2D square lattice.

Note that, in the above sequence of arguments, one step is particularly crucial, namely, the universality of the 2D cluster states: this property is used to “map” an arbitrary state $|\varphi_{\tilde{C}}\rangle$, and hence the associated partition function, to the 2D cluster state; i.e., all states can be “reduced” to this single structure.

We give a few remarks regarding this construction. In Eq. (6), note that the product state $|\gamma\rangle$ is determined by both the interaction graph G and the couplings $\{J_{ab}, h_a\}$ of the original model. On the one hand, it contains the states $|\alpha_{ab}\rangle$ and $|\alpha_a\rangle$ encoding the couplings of the original model; on the other hand, $|\gamma\rangle$ contains states $|\beta_j\rangle$ and $|0_Y\rangle$ corresponding to the sequence of one-qubit measurements which are to be implemented in order to generate $|\varphi_{\tilde{C}}\rangle$ from the universal resource $|\varphi_{\tilde{C}}\rangle$. In going from Eq. (6) to Eq. (7), the state $|\gamma\rangle$ in turn determines the

couplings in which the 2D model is to be evaluated. Note that the decorated cluster state $|\varphi_{\tilde{C}}\rangle$ has vertex qubits and edge qubits. The factors of $|\gamma\rangle$ acting on the edge qubits determine the pairwise interactions J'_{ij} , whereas the factors of $|\gamma\rangle$ acting on the vertex qubits determine the external fields h'_i . The tensor factors of $|\gamma\rangle$ which act on the edge qubits are all equal to $|0_Y\rangle \propto |0\rangle + i|1\rangle$. This implies, in particular, that, in (7), only homogeneous pairwise couplings J'_{ij} need to be considered. Furthermore, due to the imaginary unit “ i ” in $|0_Y\rangle$, these couplings generally lie in a complex parameter regime; in particular, one can show that $\beta J'_{ij} = -i\pi/4$ is a correct choice. Also, the fact that the J'_{ij} can be chosen to be homogeneous implies that all information regarding the pairwise couplings J_{ab} and external fields h_a of the original model, and the graph G of this model, will be encoded in the factors of $|\gamma\rangle$ acting on the vertex qubits, and thus in the external fields h'_i (which will typically be inhomogeneous). We further remark that the part of $|\gamma\rangle$ acting on the vertex qubits generically also corresponds to complex interaction strengths h'_i (e.g., $|\beta\rangle$ may contain Y eigenstates). A special role is played by those factors of $|\beta\rangle$ which are equal to Z eigenstate $|0\rangle \propto e^\infty|0\rangle + e^{-\infty}|1\rangle$. These states give rise to “infinitely large” external fields at the corresponding vertices, which effectively corresponds to a boundary condition.

In conclusion, the universality of the 2D cluster states $|\varphi_{\tilde{C}}\rangle$ in the context of MQC implies that the Ising partition function on any graph can be expressed as a special instance of a (polynomially enlarged) 2D Ising model with complex, homogenous pairwise interactions and complex, inhomogenous external fields. Note that even though such complex interaction strengths do not correspond to physical models, considering the partition function as a function with complex arguments is commonly done, e.g., in the context of evaluating the Tutte polynomial or finding (complex) zeros of Z_G to identify phase transition points [2].

VI. Generalizations to q -state models.—Our results can also be generalized to q -state spin models such as the Potts model [1]. We showed in Ref. [5] that the partition function of a q -state Potts model on a graph $G = (V, E)$ can be written as the overlap between a stabilizer state $|\varphi_{\tilde{C}}^q\rangle$ and a complete product state $|\chi\rangle$: $Z_G \propto \langle \chi | \varphi_{\tilde{C}}^q \rangle$. Similar to the treatment of the Ising model, the state $|\varphi_{\tilde{C}}^q\rangle$ depends only on the graph, and the state $|\chi\rangle = \otimes_{ab} |\chi_{ab}\rangle$ is a complete product state depending only on the couplings of the model. However, the main difference is that the single-particle systems are no longer qubits, but q -dimensional systems. For example, one finds $|\chi_{ab}\rangle = e^{\beta J_{ab}}|0\rangle + \sum_{k=1}^{q-1} |k\rangle$ [5]. Interestingly, the partition function of such a q -state model (for arbitrary graphs) can again be expressed as a special instance of the partition function of the 2D-Ising model (with $q = 2$) and complex parameters—again using the connection to MQC. To achieve this we use that any q -dimensional product state can be mapped

by a suitable unitary operation to a product state of $m_q = \lceil \log_2 q \rceil$ qubits: e.g., $|\chi_{ab}\rangle = U_{ab}^\dagger |0\rangle^{\otimes m_q}$. As q is fixed, the unitary U_{ab} can be prepared with a constant number of two-qubit gates. Being a stabilizer state, $|\varphi_G^q\rangle$ is preparable by a polysized (qubit) circuit. It follows that Z_G can be written as the inner product of an efficiently preparable state $|\varphi\rangle := \otimes_{ab} U_{ab} |\varphi_G^q\rangle$ (which is now regarded as a multiqubit state) with a product state $|0\rangle^{\otimes m_q |E|}$. The universality of $|\varphi_{\tilde{G}}\rangle$ for MQC now implies that $|\varphi\rangle$ can be obtained by performing single-qubit measurements on a polynomially enlarged cluster state $|\varphi_{\tilde{G}}\rangle$. In particular, Eq. (4) can be applied to $|\psi\rangle \equiv |\varphi\rangle$. Using a similar argument to Sec. , this implies that the Potts model partition function is a special instance of the partition function of a polynomially enlarged 2D-Ising model with properly tuned complex parameters and two-state spins.

The above strategy can even be applied to q -state models beyond the Potts model, e.g., to all models on directed graphs where the Hamiltonians are arbitrary functions of the difference (modulo q) between spin values, including arbitrary local terms, while still obtaining a 2D-Ising model with polynomially more spins. Even more generally, one can verify that the partition function of an arbitrary q -state spin model (with finite q), where arbitrary pairwise or even k -body interactions with bounded k are allowed, can be written as the overlap between a suitable quantum state and a product state. This immediately implies that every partition function can be expressed as a special instance of the 2D-Ising model. However, in general an exponential overhead may be required.

We further remark that the 2D square lattice does not play a special role in this context: there are many other models with a similar completeness property [11]. For example, all Ising models on a graph G whose associated graph state $|\varphi_{\tilde{G}}\rangle$ is a universal resource for MQC allows one to draw the same conclusions as for the 2D square lattice. Examples of such other universal models for MQC include, e.g., hexagonal, triangular, and kagome lattices [12], 3D lattices as well as 2D lattices with holes. On the other hand, all models corresponding to graph states $|\varphi_{\tilde{G}}\rangle$ which are not universal resources for MQC (in the sense of universal state preparation [12]) are not capable of expressing partition functions of, e.g., the 2D-Ising model (or other complete models). Examples of “noncomplete” interaction patterns are 1D structures such as chains or trees, or more generally all graphs where the decorated graph \tilde{G} has bounded rank width [12].

VII. Summary.—We have established a connection between evaluating the partition function of a general class of classical spin models and measurement-based quantum computation. We have used the universality of the 2D cluster states, in particular, the possibility of preparing any other quantum state by means of projective single-qubit measurements from a sufficiently large universal

state, to show a type of completeness of the classical 2D Ising model: the partition function of any classical spin model (Ising and Potts model on arbitrary graphs, and beyond) can be recovered as a special case of the Ising model on a sufficiently large 2D square lattice with complex couplings. Moreover, we have given an explicit, efficient construction of the corresponding 2D model.

Finally, it is an interesting open problem whether a restriction to the real (and thus physical) parameter regime of the 2D-Ising model is possible while keeping the completeness property. It would also be interesting to investigate how the explicit reductions obtained in this Letter may be related to previous results regarding the NP-completeness of the 2D Ising model [3] (see also [13]).

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