No-Ghost Theorem for the Fourth-Order Derivative Pais-Uhlenbeck Oscillator Model

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A new realization of the fourth-order derivative Pais-Uhlenbeck oscillator is constructed. This realization possesses no states of negative norm and has a real energy spectrum that is bounded below. The key to this construction is the recognition that in this realization the Hamiltonian is not Dirac Hermitian. However, the Hamiltonian is symmetric under combined space reflection P and time reversal T. The Hilbert space that is appropriate for this PT-symmetric Hamiltonian is identified and it is found to have a positive-definite inner product. Furthermore, the time-evolution operator is unitary.

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It has long been thought that field theories based on equations of motion higher than second order are unacceptable because they possess states, known as *ghosts*, which have nonpositive norm. The purpose of this Letter is to show that this is not necessarily so, and thereby to regenerate interest in higher-order quantum field theories. Higher-order theories appear in a variety of contexts [1] and are potentially of great interest.

To explain the issues involved, we review the Lee model. The model was proposed in 1954 as a trilinearly coupled quantum field theory in which the renormalization program can be carried out in closed form [2]. However, just one year later it was argued that this theory has a ghost state [3]. Specifically, a ghost appears in the Lee model when the renormalized coupling constant exceeds a critical value. Above this critical value, the Lee-model Hamiltonian becomes non-Hermitian in the Dirac sense because its trilinear interaction term acquires an imaginary coefficient. (Dirac-Hermitian conjugation is combined matrix transposition and complex conjugation.) In the non-Hermitian phase of the Lee model a state of negative Dirac norm emerges.

For the past half century, there have been many attempts to make sense of the Lee model as a valid quantum theory (starting as early as [4]), but it was not until 2005 that it was shown that it is possible to formulate the theory without a ghost [5]. The solution to the Lee-model-ghost problem is that when the coupling constant exceeds its critical value, the Hamiltonian transits from being Dirac Hermitian to being PT symmetric, i.e., symmetric under combined parity reflection and time reversal. In the sector in which the ghost appears the PT symmetry is *unbroken*; that is, all energy eigenvalues are real. For any non-Hermitian, PT-symmetric Hamiltonian in such a phase, one should not use the Dirac norm. Rather, one should introduce an alternate inner product [6–9]. With respect to the new inner product appropriate for the PT-symmetric phase of the Lee model, the Hamiltonian becomes self-adjoint and its ghost state is reinterpreted as an ordinary quantum state with positive PT norm. This same procedure has been applied to other problematic models [10].

The purpose of this Letter is to show that this same prescription can be implemented in the fourth-order-derivative Pais-Uhlenbeck (PU) oscillator model, the prototypical higher-derivative quantum field theory. We construct here a realization in which the Hamiltonian has an unbroken PT symmetry and the negative Dirac-norm states that are thought to arise are really ordinary quantum states having positive PT norm.

The action of the PU model is acceleration dependent:

$$I_{\rm PU} = \frac{\gamma}{2}\int dt[\ddot{z}^2 - (\omega_1^2 + \omega_2^2)\dot{z}^2 + \omega_1^2\omega_2^2 z^2], \qquad (1)$$

where γ, ω_1, and ω_2 are all positive constants, and without loss of generality we take $\omega_1 \geq \omega_2$ [11]. This model represents two oscillators coupled by a fourth-order equation of motion: $d^4z/dt^4 + (\omega_1^2 + \omega_2^2)d^2z/dt^2 + \omega_1^2\omega_2^2 z = 0$ [12]. With $\dot{z}$ serving as the canonical conjugate of both z and $\ddot{z}$, the system is constrained and its Hamiltonian can be found by the method of Dirac constraints. To this end, in place of $\dot{z}$ we introduce a new dynamical variable x (with corresponding conjugate p_x), and via the Dirac method we construct the Hamiltonian [13]

$$H_{\rm PU} = \frac{p_x^2}{2\gamma} + p_z x + \frac{\gamma}{2}(\omega_1^2 + \omega_2^2)x^2 - \frac{\gamma}{2}\omega_1^2\omega_2^2 z^2. \qquad (2)$$

This Hamiltonian depends on two coordinates x and z, and their canonical conjugates, p_x and p_z. The Poisson-bracket algebra of the five operators x, p_x, z, p_z, and H is closed, with nonzero brackets in the x, p_x, z, p_z sector being $\{x, p_x\} = 1$ and $\{z, p_z\} = 1$. This construction is independent of the classical equations of motion and thus holds for both stationary and nonstationary classical paths. Thus, we can use it to quantize the model, with nonzero commutators being $[x, p_x] = i$ and $[z, p_z] = i$.

Making the standard substitutions

$$\begin{aligned} z &= a_1 + a_1^\dagger + a_2 + a_2^\dagger, \\ p_z &= i\gamma\omega_1\omega_2^2(a_1 - a_1^\dagger) + i\gamma\omega_1^2\omega_2(a_2 - a_2^\dagger), \\ x &= -i\omega_1(a_1 - a_1^\dagger) - i\omega_2(a_2 - a_2^\dagger), \\ p_x &= -\gamma\omega_1^2(a_1 + a_1^\dagger) - \gamma\omega_2^2(a_2 + a_2^\dagger), \end{aligned} \tag{3}$$

we obtain a Hamiltonian and commutator algebra [13]

$$H_{\rm PU} = 2\gamma(\omega_1^2 - \omega_2^2)(\omega_1^2 a_1^\dagger a_1 - \omega_2^2 a_2^\dagger a_2) + \tfrac{1}{2}(\omega_1 + \omega_2),$$
$$\omega_1[a_1, a_1^\dagger] = -\omega_2[a_2, a_2^\dagger] = 1/[2\gamma(\omega_1^2 - \omega_2^2)] \tag{4}$$

that possess two distinct Fock-space realizations. Specifically, if we take a_1 and a_2 to annihilate the no-particle state $|\Omega\rangle$ according to $a_1|\Omega\rangle = 0$, $a_2|\Omega\rangle = 0$, the energy spectrum that ensues is then bounded below with $|\Omega\rangle$ being the ground state with energy $(\omega_1 + \omega_2)/2$. However, here the excited state $a_2^\dagger|\Omega\rangle$, which lies at energy ω_2 above the ground state, has a Dirac norm $\langle\Omega|a_2 a_2^\dagger|\Omega\rangle$ that is negative. Alternatively, if we take a_1 and $a_2^\dagger$ to annihilate the no-particle state $|\Omega\rangle$, according to $a_1|\Omega\rangle = 0$, $a_2^\dagger|\Omega\rangle = 0$, the theory would then be free of negative-norm states, but the energy spectrum would be unbounded below. Both of these realizations are undesirable and characterize the generic problems that are thought to afflict higher-derivative quantum theories.

We propose an alternative realization of the above bounded-below energy sector. To formulate it, we supply some global information and we make a standard wave-mechanics representation of the Schrödinger equation $H_{\rm PU}\psi_n = E_n\psi_n$ by setting $p_z = -i\partial/\partial z$, $p_x = -i\partial/\partial x$. In this representation the state whose energy is $(\omega_1 + \omega_2)/2$ has eigenfunction [13]

$$\psi_0(z, x) = \exp[\frac{\gamma}{2}(\omega_1 + \omega_2)(\omega_1\omega_2 z^2 - x^2) + i\gamma\omega_1\omega_2 zx],$$

and the states of higher energy have eigenfunctions that are polynomial functions of x and z times $\psi_0(z, x)$. The eigenfunction $\psi_0(z, x)$ is not normalizable on the real-z axis; it grows exponentially as $z \to \pm\infty$. Evidently, for these eigenfunctions the realization $p_z = -i\partial/\partial z$ is not Hermitian on the real-z axis. The representation of z and p_z as Dirac-Hermitian operators in (3) does not apply. Hence, the Hamiltonian and commutators of (4) do not characterize this realization and the analysis that leads to a Dirac-norm ghost in this sector is avoided [14].

Since we started with just one underlying input classical theory based on the $I_{\rm PU}$ action of (1), it cannot be that this input classical theory is the classical limit of both the bounded-below and the unbounded-below energy-sector realizations of the quantum PU theory. Rather, from the way in which theories are canonically quantized, the coordinates of the underlying classical theory correspond to the eigenvalues of quantum-mechanical operators that are Dirac Hermitian on the real axis. It is thus the unbounded-below quantum-mechanical energy sector that has the classical PU theory as its classical limit, with the bounded-below quantum-mechanical energy-sector realization corresponding to something different. [The eigenfunction of the typical unbounded-below energy-sector state with energy $(\omega_1 - \omega_2)/2$ is obtained by replacing ω_2 by $-\omega_2$ in $\psi_0(z, x)$, with $p_z = -i\partial/\partial z$ being a well-behaved operator in this sector.]

Such a situation is not unprecedented in physics. For instance, on quantizing the orbital angular momentum $\vec{L} = \vec{r} \times \vec{p}$, one finds not only integer spin representations and their associated spherical harmonics, but also half-integer spin representations. These additional representations bear no relation to any motion of coordinates in phase space and they are divorced from the underlying classical phase space whose quantization led us to uncover their existence in the first place. Hence, of the two types of realizations of the angular-momentum commutator algebra (integer and half-integer spin), only one of them has a connection to the underlying classical theory whose quantization led to the commutator algebra.

Thus, to explore the bounded-below energy sector of the quantized PU theory, we set aside the underlying classical theory with its real z and x coordinates. Instead, we begin with the Hamiltonian (2) considered as an *ab initio* input quantum-mechanical Hamiltonian subject to the commutator algebra constraints $[x, p_x] = i$, $[z, p_z] = i$. We thus seek a differential representation of this commutator algebra in some c-number parameter space, a space that we shall also label by parameters z and x. Since such a parameter space is needed to represent the commutator algebra, we can just as easily use complex parameters as real ones in quantum mechanics as long as they correctly implement the commutator algebra [15].

Given the structure of $\psi_0(z, x)$, and recalling that one can only use the realization $p_z = -i\partial/\partial z$ when the $[z, p_z]$ commutator acts on test functions that are well behaved, we see that such well-behaved functions cannot be taken to lie on the real-z axis of the parameter space. Moreover, they cannot even lie in two *Stokes wedges* of angular opening 90° and centered about the positive- and negative-real axes (the east and west quadrants of the letter X) in the complex-z plane. However, in the complementary 90° Stokes wedges centered about the positive- and negative-imaginary z axes (the north and south quadrants of the letter X), $\psi_0(z, x)$ vanishes exponentially rapidly as $|z| \to \infty$. We thus restrict the Schrödinger equation eigenvalue problem to the complementary (north-south) Stokes wedges. In these wedges $\psi_0(z, x)$ is the fully normalizable ground state of the system and the energy spectrum is precisely the purely real one associated with the bounded-below energy sector.

To avoid having to work on the imaginary axis, we instead perform the (isospectral) operator similarity transform $y = e^{\pi p_z z/2} z e^{-\pi p_z z/2} = -iz$, $q = e^{\pi p_z z/2} p_z e^{-\pi p_z z/2} = ip_z$ with $[y, q] = i$. In terms of y and q the Hamiltonian takes the form

$$H = \frac{p^2}{2\gamma} - iqx + \frac{\gamma}{2}(\omega_1^2 + \omega_2^2)x^2 + \frac{\gamma}{2}\omega_1^2\omega_2^2 y^2, \qquad (5)$$

where, for simplicity, we have replaced p_x by p. In (5) the operators x, p, y, and q are now formally Dirac Hermitian on the real x and y axes, but because of the $-iqx$ term, H has become complex and is manifestly not Dirac Hermitian [16]. This non-Hermiticity is not apparent in the original form of the Hamiltonian in (2). This surprising and unexpected emergence of a non-Hermitian term in the PU Hamiltonian is the root cause of the ghost problem of the Pais-Uhlenbeck model.

While the Hamiltonian (5) is not Dirac Hermitian, it falls into a class of equally physically viable Hamiltonians, those that are PT symmetric. To demonstrate the PT symmetry we make the following assignments: Under P and T, we take p and x to transform like conventional coordinate and momentum variables. However, we define q and y to transform unconventionally in a way that has not been seen in previous studies of PT quantum mechanics; in the language of quantum field theory, q and y transform as parity scalars instead of pseudoscalars, and they have abnormal behavior under time reversal. To summarize these transformation properties, under P reflection x and p change sign but y and q do not; under $\mathcal{T}$ reflection y and p change sign but x and q do not. Thus, under combined PT reflection x and y change sign but p and q do not.

Because the PT-invariant H has an entirely real spectrum, we can introduce a positive-definite inner product and we can reinterpret the ghosts as conventional quantum states of positive PT norm. Following the standard procedures of PT quantum mechanics, we construct the PT norm by introducing an operator called the $\mathcal{C}$ *operator* [17]. The $\mathcal{C}$ operator associated with the Hamiltonian (5) satisfies three conditions:

$$\mathcal{C}^2 = 1, \quad [\mathcal{C}, PT] = 0, \quad [\mathcal{C}, H] = 0. \qquad (6)$$

The first two are kinematical, while the third is dynamical because it involves the specific Hamiltonian H.

In previous investigations it was established that $\mathcal{C}$ has the general form $\mathcal{C} = e^{\mathcal{Q}}P$, where $\mathcal{Q}$ is a real function of the dynamical variables and is Hermitian in the Dirac sense. It had been found that $\mathcal{Q}$ was odd under a change in sign of the momentum variables and even under a change in sign of the coordinate variables. However, because of the abnormal P, T behaviors of the y and q operators, the exact solution to the three simultaneous algebraic equations in (6) gives an unusual and previously unencountered bilinear structure for $\mathcal{Q}$:

$$\mathcal{Q} = \alpha pq + \beta xy, \qquad (7)$$

where $\beta = \gamma^2\omega_1^2\omega_2^2\alpha$ and $\sinh(\sqrt{\alpha\beta}) = 2\omega_1\omega_2/(\omega_1^2 - \omega_2^2)$.

Even though the form of $\mathcal{Q}$ in (7) is unprecedented in PT quantum mechanics, the effect of performing a similarity transformation on the dynamical variables x, p, y, q using $e^{\mathcal{Q}}$ still generates a canonical transformation that preserves the commutation relations, just as in previous studies. For the PU model the transformation is

$$\begin{aligned} e^{\mathcal{Q}}xe^{-\mathcal{Q}} &= xc - idqs, \quad e^{\mathcal{Q}}qe^{-\mathcal{Q}} = qc + ixs/d, \\ e^{\mathcal{Q}}ye^{-\mathcal{Q}} &= yc - idps, \quad e^{\mathcal{Q}}pe^{-\mathcal{Q}} = pc + iys/d, \end{aligned} \qquad (8)$$

where $c = \cosh(\sqrt{\alpha\beta})$, $s = \sinh(\sqrt{\alpha\beta})$, and $d = \sqrt{\alpha/\beta}$.

In PT quantum mechanics a similarity transformation on the PT-symmetric Hamiltonian with $e^{-\mathcal{Q}}/2$ yields a positive-definite Hamiltonian that is Hermitian in the Dirac sense [18]. Here, we obtain

$$\begin{aligned} \tilde{H} &= e^{-\mathcal{Q}/2}He^{\mathcal{Q}/2} \\ &= \frac{p^2}{2\gamma} + \frac{q^2}{2\gamma\omega_1^2} + \frac{\gamma}{2}\omega_1^2x^2 + \frac{\gamma}{2}\omega_1^2\omega_2^2y^2. \end{aligned}$$

The spectrum of $\tilde{H}$ is manifestly real and positive and its position and momentum operators are conventionally Hermitian. This Hamiltonian is related to the original PU Hamiltonian by a similarity transformation, which is isospectral. Thus, despite the $-iqx$ term, the positivity of the PU Hamiltonian is proved [19].

Furthermore, the eigenstates $|\tilde{n}\rangle$ of $\tilde{H}$ have positive inner product and can be normalized in the conventional Dirac way using the standard inner product $\langle\tilde{n}|\tilde{n}\rangle = 1$, where the bra vector is the Dirac-Hermitian adjoint of the ket vector. Equivalently, for the eigenstates $|n\rangle$ of the Hamiltonian H, because the vectors are mapped by $|\tilde{n}\rangle = e^{-\mathcal{Q}/2}|n\rangle$, the eigenstates of H are normalized as

$$\langle n|e^{-\mathcal{Q}}|m\rangle = \delta(m,n), \qquad \sum |n\rangle\langle n|e^{-\mathcal{Q}} = 1. \qquad (9)$$

Thus, the norm of (9) is the one that is relevant, with $\langle n|e^{-\mathcal{Q}}$ rather than $\langle n|$ being the appropriate conjugate for $|n\rangle$ [20]. Moreover, because the norm in (9) is positive and because $[H, \mathcal{C}PT] = 0$, the Hamiltonian H generates unitary time evolution, just as one would want [21], [22].

To conclude, in order to construct an acceptable realization for the Pais-Uhlenbeck oscillator, we have found the region in parameter space where operators such as $-i\partial/\partial_z$ are well defined. The appearance of a ghost state when one takes the derivative operator to be Dirac-Hermitian on the real-z axis is not an indication that there is anything wrong with the theory itself, but only with that particular realization of it. Hence, higher-derivative field theories may not be as problematic as they are often thought to be.

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[12] With the identifications $\omega_i = (\bar{k}^2 + M_i^2)^{1/2}$ $(i = 1, 2)$ this equation of motion is the quantum-mechanical limit of the field-theoretic second-plus-fourth order $(-\partial_t^2 + \nabla^2 - M_1^2)(-\partial_t^2 + \nabla^2 - M_2^2)\phi(\bar{x}, t) = 0$ in field configurations of the form $\phi(\bar{x}, t) = z(t)e^{i\bar{k}\cdot\bar{x}}$. The $M_1 = M_2 = 0$ pure fourth-order theory corresponds to $\omega_1 = \omega_2 = |\bar{k}|$.
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[14] All eigenfunctions have infinite Dirac norm: $\langle\Omega|\Omega\rangle = \int dzdx\langle\Omega|z, x\rangle\langle z, x|\Omega\rangle = \int dzdx\psi_0^c(z, x)\psi_0(z, x)$ diverges. Thus, there is no completeness relation of the form $\sum \eta_n|n\rangle\langle n| = 1$ with $\eta_n = \pm 1$ and one cannot use such a completeness relation to claim loss of unitarity.
[15] One can represent the action of the $[z, p_z]$ commutator by $[z, -i\partial_z]\psi(z) = i\psi(z)$, or equally well by $[\tilde{z}, -i\partial_{\tilde{z}}]\psi(\tilde{z}) = i\psi(\tilde{z})$, where $\tilde{z} = ze^{i\theta}$. We thus seek those regions in θ space where $\psi(ze^{i\theta})$ is well behaved.
[16] Although quantum mechanics is usually formulated in terms of Hermitian operators, Hermiticity is only sufficient to give real eigenvalues but not necessary. Specifically, while a Hermitian operator has real eigenvalues, there is no converse theorem that says that the eigenvalues of a non-Hermitian operator are not real. The Hamiltonian (5) is an example of a non-Hermitian operator that possesses a strictly real spectrum of eigenvalues.
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[19] While Stokes wedges and PT quantum mechanics are central to deriving $\tilde{H}$, via the similarity transformations $S_1 = e^{\pi p_z z/2}$ and $S_2 = e^{-\mathcal{Q}/2}$ one can go directly from (2) to (5) to $\tilde{H}$ entirely at the operator level.
[20] In an eigenvalue equation of the form $H|\psi\rangle = E|\psi\rangle$, the Hamiltonian acts linearly on ket vectors, without any reference to bra vectors. Consequently, the eigenvalue equation is unaffected if the bra vector is not taken to be the Dirac conjugate $\langle\psi|$ of the ket $|\psi\rangle$. While it has been noted [6] that in theories with non-Hermitian Hamiltonians the choice of norm may not always be unique, for our purposes here it suffices to show that in the PU theory one can explicitly construct an appropriate norm, viz., that of (9), for which there are no ghosts.
[21] Noting that the operator $\mathcal{Q}$ in (7) becomes singular as $\omega_1 \to \omega_2$, the analysis presented here only holds in the unequal-frequency case. The equal-frequency case requires special treatment and in a separate publication we will show that in this limit the unitarity of the PU oscillator is maintained.
[22] Despite the presence of the $-iqx$ term, the Heisenberg equations of motion associated with (5) yield operator quantum equations of motion of the same real form as those of the classical PU theory: $d^4A/dt^4 + (\omega_1^2 + \omega_2^2)d^2A/dt^2 + \omega_1^2\omega_2^2 A = 0$ $(A = x, p, y, q)$. With the operators x, p, y, and q all being Dirac Hermitian, we thus see that when our analysis is extended to field theory, the fields of the associated classical limit would all be real.