Regular-to-Chaotic Tunneling Rates Using a Fictitious Integrable System

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We derive a formula predicting dynamical tunneling rates from regular states to the chaotic sea in systems with a mixed phase space. Our approach is based on the introduction of a fictitious integrable system that resembles the regular dynamics within the island. For the standard map and other kicked systems we find agreement with numerical results for all regular states in a regime where resonance-assisted tunneling is not relevant.

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Tunneling of a quantum particle is one of the central manifestations of quantum mechanics. For simple 1D systems tunneling under a potential barrier is well understood and described, e.g., by using semiclassical WKB theory or the instanton approach [1]. For higher-dimensional systems so-called "dynamical tunneling" [2] occurs between regions that are separated by dynamically generated barriers. Typically, such systems have a mixed phase space in which regions of regular motion and irregular dynamics coexist. Tunneling in these systems is barely understood as it generically cannot be reduced to the instanton or WKB approach. It has been studied theoretically [3–14] and experimentally, e.g., in cold atom systems [16,17] and semiconductor nanostructures [18]. A precise knowledge of tunneling rates is of current interest for, e.g., eigenstates affected by flooding of regular islands [19,20], emission properties of optical microcavities [21], and spectral statistics in systems with a mixed phase space [22].

There are different approaches for the prediction of tunneling rates depending on the ratio of Planck's constant h to the size A of the regular island. In the semiclassical regime, $h \ll A$, small resonance chains inside the island dominate the tunneling process ("resonance-assisted tunneling") [11,12]. In contrast, we focus on the experimentally relevant regime of large h (while still $h < A$), where small resonance chains are expected to have no influence on the tunneling rates. This regime has been investigated in Ref. [14]; however, the prediction does not seem to be generally applicable (see below). Other studies in this regime investigate situations [13,23] where dynamical tunneling can be described by 1D tunneling under a barrier; however, in our opinion they are nongeneric. A generally applicable theoretical description of dynamical tunneling rates in systems with a mixed phase space is still an open question.

In this Letter we present a new approach to dynamical tunneling from a regular island to the chaotic sea. The central idea is the use of a fictitious integrable system resembling the regular island. This leads to a tunneling formula involving properties of this integrable system as well as its difference to the mixed system under consideration. It allows for the prediction of tunneling rates from any quantized torus within the regular island. We find excellent agreement with numerical data (see Fig. 1) for an example system where tunneling is not affected by phase-space structures like cantori at the border of the island. The applicability to more general systems is demonstrated for the standard map; see Fig. 4 below.

We consider 2D maps with one major regular island embedded in the chaotic region (Fig. 1, insets), which are described quantum mechanically by a unitary operator U [24]. Classically the regular and chaotic region are separated, however quantum mechanically they are coupled. This coupling has consequences for the eigenstates of U. While they are mainly regular or chaotic, i.e., concentrated on a torus inside the regular region or spread out over the chaotic sea, they do have at least a small component in the other region. This is most clearly seen for hybrid states [Fig. 2(d)]. For a wave packet started on the mth quantized torus ($m = 0, 1, \ldots, m_{\max} - 1$) coupled to an infinite chaotic sea, the decay $e^{-\gamma_m t}$ is described by a tunneling rate γ_m. For systems with a finite phase space this exponential decay occurs at most up to the Heisenberg time $\tau_H = h/\Delta_{\mathrm{ch}}$, where Δ_{ch} is the mean level spacing of the chaotic

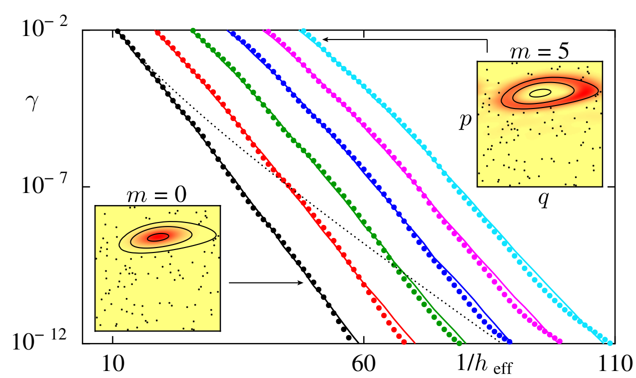

FIG. 1 (color online). Dynamical tunneling rates from a regular island to the chaotic sea for the kicked system [27]: Numerical results (dots) and prediction following from Eq. (3) (lines) vs $1/\hbar_{\mathrm{eff}}$ for quantum numbers $m \leq 5$. The insets show Husimi representations of the regular states $m = 0$ and $m = 5$ at $1/h_{\mathrm{eff}} = 50$. The prediction of Ref. [14] for $m = 0$ with a fitted prefactor is shown (dotted line).

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states. Introducing purely regular states $|\tilde{\psi}_{\rm reg}\rangle$ and orthogonal chaotic states $|\tilde{\psi}_{\rm ch}\rangle$, the tunneling rate from such a purely regular state can be expressed by Fermi's golden rule $\gamma = (2\pi/\hbar)\overline{|V|^2}\rho_{\rm ch}$, where $\rho_{\rm ch} = 1/\Delta_{\rm ch} \propto N_{\rm ch}$ is the chaotic density of states and $V = \langle\tilde{\psi}_{\rm ch}|\hat{H}|\tilde{\psi}_{\rm reg}\rangle$ for a time-independent $\hat{H}$. For a map U one replaces the local average over matrix elements in Fermi's golden rule by an average over all $N_{\rm ch}$ chaotic states and expresses γ with respect to the time period of U, yielding

$$\gamma = \sum_{\rm ch}|v|^2, \tag{1}$$

where $v = \langle\tilde{\psi}_{\rm ch}|U|\tilde{\psi}_{\rm reg}\rangle$. The eigenstates of U cannot be used for determining the small matrix elements v, as they are neither purely regular nor purely chaotic.

In order to construct purely regular and chaotic states, we introduce fictitious regular and chaotic quantum maps $U_{\rm reg}$ and $U_{\rm ch}$ [25]. Here $U_{\rm reg}$ is regular in the sense that it can be written as $e^{-i\hat{H}_{\rm reg}/\hbar_{\rm eff}}$, where $H_{\rm reg}$ is a 1D Hamiltonian, which is integrable by definition and $\hbar_{\rm eff}$ is the effective Planck constant. $H_{\rm reg}$ has to be chosen such that its dynamics over one time unit resembles the classical motion corresponding to U within the regular island as closely as possible [Fig. 2(b)]. The eigenstates $|\psi_{\rm reg}\rangle$ of $U_{\rm reg}$ are localized in the regular region and continue to decay into the chaotic sea [Fig. 2(e)]. This is the decisive property of $|\psi_{\rm reg}\rangle$, which is in contrast to those eigenstates of U that are predominantly regular but all have a small chaotic admixture. The eigenstates $|\psi_{\rm ch}\rangle$ of $U_{\rm ch}$ live in the chaotic region of U and decay into the regular island [Fig. 2(f)].

As $|\psi_{\rm reg}\rangle$ and $|\psi_{\rm ch}\rangle$ are eigenstates of different operators $U_{\rm reg}$ and $U_{\rm ch}$, they are not necessarily orthogonal, $\langle\psi_{\rm ch}|\psi_{\rm reg}\rangle = \chi$ with $0 \leq |\chi| \ll 1$. In order to apply Fermi's golden rule we introduce orthonormalized states

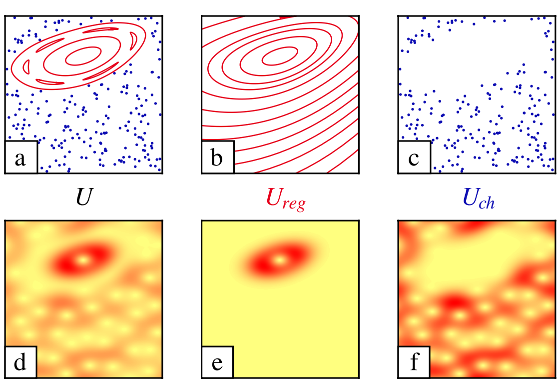

FIG. 2 (color online). (a)–(c) The classical phase space corresponding to some quantum maps U, $U_{\rm reg}$, and $U_{\rm ch}$. (d)–(f) Husimi representation of eigenstates of such maps. Eigenstates of U have a regular and a chaotic component, as illustrated in the strongest form of a hybrid state (d). Eigenstates $|\psi_{\rm reg}\rangle$ ($|\psi_{\rm ch}\rangle$) of $U_{\rm reg}$ ($U_{\rm ch}$) are purely regular (chaotic).

$|\tilde{\psi}_{\rm reg}\rangle = |\psi_{\rm reg}\rangle$, $\quad |\tilde{\psi}_{\rm ch}\rangle = (|\psi_{\rm ch}\rangle - \chi^*|\psi_{\rm reg}\rangle)/\sqrt{1-|\chi|^2}$, leading to $\langle\tilde{\psi}_{\rm ch}|\tilde{\psi}_{\rm reg}\rangle = 0$. We find up to first order in χ for the coupling matrix element

$$v \approx \langle\psi_{\rm ch}|U - U_{\rm reg}|\psi_{\rm reg}\rangle, \tag{2}$$

which can be inserted into Eq. (1). The appearing term $\sum_{\rm ch}|\psi_{\rm ch}\rangle\langle\psi_{\rm ch}|$ is semiclassically equal to the projection operator onto the chaotic region. It can be approximated as $\mathbb{1} - P_{\rm reg}$, where $P_{\rm reg}$ is a projector onto the regular island. This yields

$$\gamma_m \approx \|(\mathbb{1} - P_{\rm reg})(U - U_{\rm reg})|\psi_{\rm reg}^m\rangle\|^2 \tag{3}$$

as our main result, which involves properties of the fictitious regular system $U_{\rm reg}$ and the difference $U - U_{\rm reg}$. It allows for determining tunneling rates from the regular state on the mth quantized torus to the chaotic sea.

The most difficult step in applying Eq. (3) is the determination of the fictitious integrable system $U_{\rm reg}$, defined by a time-independent 1D Hamiltonian $H_{\rm reg}(p, q)$. On the one hand, its dynamics over one time unit should resemble the classical motion corresponding to U within the regular island as closely as possible. As a result, the contour lines of $H_{\rm reg}(p, q)$ in phase space [Fig. 2(b)] approximate the KAM curves of the classical map [Fig. 2(a)]. On the other hand, the function $H_{\rm reg}(p, q)$ should extrapolate sufficiently smoothly to the remaining phase-space region. This is essential for the quantum eigenstates of $H_{\rm reg}$ to have reasonable tunneling tails in the neighborhood of the regular island. Finding an optimal $H_{\rm reg}$ is a difficult task. In fact, it will resemble the dynamics within the island with finite accuracy only, due to the generic presence of small resonance chains and the complicated structure of tori at the boundary of a regular island. Similar problems appear for the analytic continuation of a regular torus into complex space due to the existence of a so-called natural boundary [4,8,10,11]. For the quantum tunneling problem at not too small $h_{\rm eff}$ and thus for a finite phase-space resolution, however, such an $H_{\rm reg}$ with limited accuracy can be good enough. We will discuss below two approaches [26] leading to a sufficiently good $H_{\rm reg}$ for the prediction of tunneling rates. Quantizing $H_{\rm reg}$ yields the required quantum mechanical operator $U_{\rm reg} = e^{-i\hat{H}_{\rm reg}/\hbar_{\rm eff}}$ with corresponding eigenfunctions $|\psi_{\rm reg}^m\rangle$. For the numerical evaluation of Eq. (3) in Fig. 1, it is convenient to replace $U_{\rm reg}|\psi_{\rm reg}^m\rangle$ by $e^{-iE_m/\hbar_{\rm eff}}|\psi_{\rm reg}^m\rangle$ and approximate $P_{\rm reg} \approx \sum|\psi_{\rm reg}^m\rangle\langle\psi_{\rm reg}^m|$, where the sum extends over $m = 0, 1, \ldots, \lfloor A/h_{\rm eff} - 1/2\rfloor$.

In the following we will discuss the application of Eq. (3) for 1D kicked systems $H(p, q, t) = T(p) + V(q)\sum_n \delta(t - n)$, yielding the classical mapping: $q_{t+1} = q_t + T'(p_t)$, $p_{t+1} = p_t - V'(q_{t+1})$. The corresponding quantum map over one kick period is $U = \exp[-iV(\hat{q})/\hbar_{\rm eff}]\exp[-iT(\hat{p})/\hbar_{\rm eff}] = U_V U_T$, where $\hbar_{\rm eff}$ is the ratio of Planck's constant h to the area of a phase-space unit cell. We consider a compact phase space with periodic boundary conditions for $q \in [-1/2, 1/2]$ and $p \in [-1/2, 1/2]$. In order to avoid the influence of resonances and cantori on the tunneling rates, we use a system containing one regular island with very small resonance chains and a narrow transition region to a homogeneous chaotic sea. It is obtained by an appropriate choice of the functions $V'(q)$ and $T'(p)$ [27]. The phase space is shown in the Husimi function insets of Fig. 1. After determining U_{reg} and $|\psi_{\text{reg}}^m\rangle$ as described in the last paragraph, we predict tunneling rates by evaluating Eq. (3). Figure 1 shows a comparison to tunneling rates, determined numerically by absorbing boundary conditions at $q = \pm 1/2$ and taking twice the distance between the eigenvalue of the mth regular state and the unit circle. We find excellent agreement for the tunneling rates γ_m over 10 orders of magnitude. The deviations for the smallest γ can be attributed to the beginning of the resonance-assisted tunneling regime. We determine H_{reg} using the Lie-transformation method [28]. With increasing N, the tunneling rates following from Eq. (3) converge to a constant value [see Fig. 3(a)] and we choose $N = 10$ for the predictions in Fig. 1. Note that for sufficiently high N [not shown in Fig. 3(a)] H_{reg} and the prediction for γ are expected to diverge.

We now demonstrate that an analytical evaluation of Eq. (3) is possible for our example system. We define functions $\tilde{V}(q)$ and $\tilde{T}(p)$ by a low order Taylor expansion of $V(q)$ and $T(p)$, respectively, around the center of the regular island [29]. This results in a unitary operator $U_{\tilde{V}} U_{\tilde{T}}$ with the following properties: (i) The corresponding classical dynamics is not necessarily regular. (ii) It is close, however, to a regular quantum map U_{reg} beyond the border of the island and can therefore be used in Eq. (3) instead of U_{reg}. (iii) Within the island it has an almost identical classical dynamics as U. Therefore $(U - U_{\tilde{V}} U_{\tilde{T}})|\psi_{\text{reg}}\rangle$ has almost all of its weight in the chaotic region and the projection operator $\mathbb{1} - P_{\text{reg}}$ can be neglected in Eq. (3). With the definitions $\mathbb{1} + \varepsilon_V \equiv e^{-(i/\hbar_{\text{eff}})[V(\hat{q}) - \tilde{V}(\hat{q})]}$ and $\mathbb{1} + \varepsilon_T \equiv e^{-(i/\hbar_{\text{eff}})[T(\hat{p}) - \tilde{T}(\hat{p})]}$ one obtains $\gamma_m = \| U_{\tilde{V}}[\varepsilon_V + \varepsilon_T + \varepsilon_V \varepsilon_T] U_{\tilde{T}} |\psi_{\text{reg}}^m\rangle \|^2$. We find that typically the third contribution is negligible, leading to

$$\gamma_m \approx 2 \int dq |\psi_{\text{reg}}^m(q)|^2 \left(1 - \cos\frac{V(q) - \tilde{V}(q)}{\hbar_{\text{eff}}}\right) + 2 \int dp |\psi_{\text{reg}}^m(p)|^2 \left(1 - \cos\frac{T(p) - \tilde{T}(p)}{\hbar_{\text{eff}}}\right). \quad (4)$$

In the last step the sums over the discrete position and momentum values have been replaced by integrals, which is valid in the semiclassical limit. Agreement with the direct evaluation of Eq. (3) is found (not shown). If an analytical WKB expression for the regular states $|\psi_{\text{reg}}^m\rangle$ is known, Eq. (4) can be evaluated further. This is the case for a different parameter set [30], which yields a tilted harmonic oscillator-like island embedded in a chaotic sea. We approximate $V(q) - \tilde{V}(q)$ and $T(p) - \tilde{T}(p)$ by linear functions and use a WKB ansatz for the regular wave function.

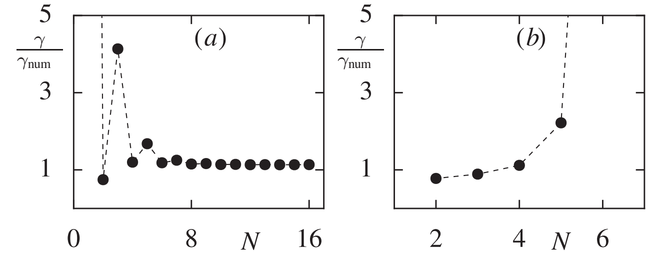

FIG. 3. Predicted tunneling rate, Eq. (3), normalized by the numerical value for $m = 0$, $h_{\text{eff}} = 1/32$ vs order N of H_{reg} corresponding to (a) Fig. 1 ($N = 10$) and (b) Fig. 4 ($N = 4$).

It turns out that the integral is proportional to the square of the modulus of the regular wave function at the border of the regular island. We obtain

$$\gamma_m = c \frac{h_{\text{eff}}}{\beta_m} \exp\left(-\frac{2A}{h_{\text{eff}}}\left[\beta_m - \alpha_m \ln\left(\frac{1 + \beta_m}{\sqrt{\alpha_m}}\right)\right]\right) \quad (5)$$

as the semiclassical prediction for the tunneling rate of the mth regular state, where $\alpha_m = (m + 1/2)(A/h_{\text{eff}})^{-1}$, $\beta_m = \sqrt{1 - \alpha_m}$, $A \approx 0.28$ is the area of the regular island and $c \approx 1$ is h_{eff} independent by a rough semiclassical estimate. The prediction, Eq. (5), gives excellent agreement with numerically determined data over 10 orders of magnitude in γ (not shown). Let us make the following remarks concerning Eq. (5): (i) The only information about this nongeneric island with constant rotation number is A/h_{eff} as in Ref. [14]. (ii) While the term in square brackets semiclassically approaches 1, it is relevant for large h_{eff}. (iii) In contrast to Eq. (4), where the chaotic properties are contained in the differences $V(q) - \tilde{V}(q)$ and $T(p) - \tilde{T}(p)$, they appear in the prefactor c via the linear approximation of these differences.

The paradigmatic model of quantum chaos is the standard map $[T(p) = p^2/2,\ V(q) = -K/(4\pi^2)\cos(2\pi q)]$, which for $K = 2.9$ has a large generic regular island. Absorbing boundary conditions at $q = \pm 1/2$ lead to strong fluctuations of the numerically determined tunneling rates as a function of h_{eff}, presumably due to cantori. When choosing $q = \pm 1/4$, which is closer to the island, we find smoothly decaying tunneling rates (dots in Fig. 4). Evaluating Eq. (3) gives good agreement with these numerical data. Note that this is the first quantitative prediction of regular-to-chaotic tunneling rates for the standard map. Here we determine H_{reg} by first using the frequency map analysis [31] for characterizing the properties of the regular island. This information is used to find the optimal 2D Fourier series of order N for H_{reg}. The tunneling rates following from Eq. (3) show for increasing N the expected divergence [see Fig. 3(b)]. For the predictions in Fig. 4 we choose $N = 4$ as the largest order before this divergence.

We now want to discuss the relation of our approach to previous studies. The semiclassical formula presented in Ref. [14] (dotted lines in Figs. 1 and 4) deviates from numerically determined tunneling rates. It works best for

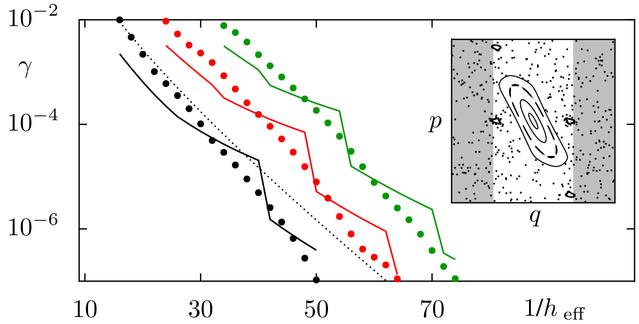

FIG. 4 (color online). Tunneling rates for the standard map ($K = 2.9$) for $m \leq 2$. Prediction of Eq. (3) (lines) and numerical results (dots), obtained using an absorbing boundary at $q = \pm 1/4$ (gray-shaded area of the inset). Prediction of Ref. [14] for $m = 0$ with a fitted prefactor (dotted line).

the case of constant rotation number, which according to Ref. [15] is the approximation used in Ref. [14]. However, it seems to be not generally applicable. The system studied in Ref. [13] can be approximated by a 1D Hamiltonian $H_{\rm reg}(p, q)$ with a cubic potential. Here the tunneling path ends far away from the island. In such a case our result is also applicable but the use of the WKB expression presented in Ref. [13] is more convenient. In general situations, however, the main contribution comes from tunneling to the neighborhood of the regular island as seen, e.g., from Eq. (4). We also performed successful tests on the tunneling system investigated in Ref. [32].

In summary, we have derived a quantum mechanical formula Eq. (3) for the tunneling rates, which involves the fictitious integrable system $U_{\rm reg}$ and the difference $U - U_{\rm reg}$. It is the basis for deriving semiclassical expressions, which we demonstrated with Eqs. (4) and (5) for the case of a fictitious regular system, that is well approximated by a kicked system. Still there are open questions about dynamical tunneling from a regular island to the chaotic sea: (i) Which properties of the regular island (e.g., size, winding number, shape) and which properties of the chaotic sea are relevant in general? (ii) Can the approach be combined with the resonance-assisted tunneling description and how can cantori be accounted for? (iii) How can it be generalized to time-independent Hamiltonian systems, in particular, billiards? We hope that our approach with a fictitious integrable system will allow us to answer these questions.

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[1] E. Merzbacher, *Quantum Mechanics* (Wiley, New York, 1998); E. Gildener and A. Patrascioiu, Phys. Rev. D **16**, 423 (1977).

[2] M. J. Davis and E. J. Heller, J. Chem. Phys. **75**, 246 (1981).

[3] J. D. Hanson, E. Ott, and T. M. Antonsen, Phys. Rev. A **29**, 819 (1984).

[4] M. Wilkinson, Physica (Amsterdam) **21D**, 341 (1986).

[5] O. Bohigas, S. Tomsovic, and D. Ullmo, Phys. Rep. **223**, 43 (1993).

[6] S. Tomsovic and D. Ullmo, Phys. Rev. E **50**, 145 (1994).

[7] S. Tomsovic, J. Phys. A **31**, 9469 (1998).

[8] S. C. Creagh, *Tunneling in Complex Systems* (World Scientific, Singapore, 1998).

[9] S. D. Frischat and E. Doron, Phys. Rev. E **57**, 1421 (1998).

[10] A. Shudo and K. S. Ikeda, Phys. Rev. Lett. **74**, 682 (1995); Physica (Amsterdam) **115D**, 234 (1998); T. Onishi, A. Shudo, K. S. Ikeda, and K. Takahashi, Phys. Rev. E **64**, 025201(R) (2001).

[11] O. Brodier, P. Schlagheck, and D. Ullmo, Phys. Rev. Lett. **87**, 064101 (2001); Ann. Phys. (N.Y.) **300**, 88 (2002); C. Eltschka and P. Schlagheck, Phys. Rev. Lett. **94**, 014101 (2005).

[12] S. Wimberger, P. Schlagheck, C. Eltschka, and A. Buchleitner, Phys. Rev. Lett. **97**, 043001 (2006).

[13] M. Sheinman, S. Fishman, I. Guarneri, and L. Rebuzzini, Phys. Rev. A **73**, 052110 (2006).

[14] V. A. Podolskiy and E. E. Narimanov, Phys. Rev. Lett. **91**, 263601 (2003); see Ref. [15] for a corrected formula, which is used in the figures.

[15] M. Sheinman, Masters thesis, Technion, 2005, Eq. (A.14).

[16] D. A. Steck, W. H. Oskay, and M. G. Raizen, Science **293**, 274 (2001).

[17] W. K. Hensinger *et al.*, Nature (London) **412**, 52 (2001).

[18] T. M. Fromhold *et al.*, Phys. Rev. B **65**, 155312 (2002).

[19] H. Schanz, M.-F. Otto, R. Ketzmerick, and T. Dittrich, Phys. Rev. Lett. **87**, 070601 (2001).

[20] A. Bäcker, R. Ketzmerick, and A. G. Monastra, Phys. Rev. Lett. **94**, 054102 (2005); Phys. Rev. E **75**, 066204 (2007).

[21] J. Wiersig and M. Hentschel, Phys. Rev. A **73**, 031802(R) (2006).

[22] M. V. Berry and M. Robnik, J. Phys. A **17**, 2413 (1984).

[23] J. Feist, A. Bäcker, R. Ketzmerick, S. Rotter, B. Huckestein, and J. Burgdörfer, Phys. Rev. Lett. **97**, 116804 (2006).

[24] M. V. Berry, N. L. Balazs, M. Tabor, and A. Voros, Ann. Phys. (N.Y.) **122**, 26 (1979).

[25] A related idea was presented in [5].

[26] For a different approach see R. Scharf, J. Phys. A **21**, 4133 (1988), and references therein.

[27] We start with functions $t'(p) = 1/2 \pm (1 - 2p)$ for $0 < \pm p < 1/2$ and $v'(q) = -rq + Rq^2$ for $-1/2 < q < 1/2$. Smoothing the periodically extended functions with a Gaussian, $G(z) = \exp(-z^2/2\varepsilon^2)/\sqrt{2\pi\varepsilon^2}$, gives analytic functions $T'(p) = \int dz t'(p + z)G(z)$ and $V'(q) = \int dz v'(q + z)G(z)$. We take $r = 0.26$, $R = 0.4$, and $\varepsilon = 0.005$. A similar model, restricted to constant rotation number ($R = 0$), has been used in [19,20].

[28] A. J. Lichtenberg and M. A. Liebermann, *Regular and Stochastic Motion* (Springer-Verlag, New York, 1983).

[29] For the map defined in [27] we get $\tilde{T}'(p) = 3/2 - 2p$ and $\tilde{V}'(q) = -rq + R(q^2 + \varepsilon^2)$.

[30] We use the map defined in [27] with parameters $r = 0.65$, $R = 0$, and $\varepsilon = 0.002$.

[31] J. Laskar, C. Froeschlé, and A. Celletti, Physica (Amsterdam) **56D**, 253 (1992).

[32] A. Ishikawa, A. Tanaka, and A. Shudo, J. Phys. A **40**, F397 (2007).